Outline of Vanderbilt Talk

① $\boxed{\text{Motivation}}$: Classify Heisenberg pairs (A,B) on a Hilbert space.

Theorem: If G is a locally cmpt abelian group, every **Heisenberg rep'n** is unitarily equivalent to (a direct sum of copies of) the **Schrödinger rep'n**.

Heisenberg rep'n: (H, R, S)

$$R: G \longrightarrow \mathcal{U}(H) \quad \text{strongly cts. gp rep'n}$$
$$S: \hat{G} \longrightarrow \mathcal{U}(H)$$

$$S_\gamma R_x = \gamma(x) R_x S_\gamma \quad \forall x \in G,\ \gamma \in \hat{G}$$

Schrödinger rep'n: $(L^2(G), U, V)$

$$U: G \longrightarrow \mathcal{U}(L^2(G))$$
$$x \longmapsto U_x, \quad (U_x f)(y) = f(x^{-1}y)$$
$$V: \hat{G} \longrightarrow \mathcal{U}(L^2(G))$$
$$\gamma \longmapsto V_\gamma \quad (V_\gamma f)(y) = \gamma(y) f(y).$$

Unitarily equ: $\exists\, W: \bigoplus_{j\in J} L^2(G) \longrightarrow H$ such that

$$R = \mathrm{ad}_W \circ \Big(\bigoplus_{j\in J} U\Big) \quad \text{and} \quad S = \mathrm{ad}_W \circ \Big(\bigoplus_{j\in J} V\Big)$$

Corollary: If $G = \mathbb{R}$, $R = \{e^{ixB}\}_{x\in\mathbb{R}}$, $S = \{e^{i\gamma A}\}_{\gamma\in\mathbb{R}}$ for self-adjoint op. A, B on H, and $U = \{e^{ixD}\}_{x\in\mathbb{R}}$, $V = \{e^{i\gamma M}\}_{\gamma\in\mathbb{R}}$ where $Df = -if'$ $\forall f \in \mathrm{dom}\, D \subsetneq L^2(\mathbb{R})$ and $[Mf](x) = xf(x)$ $\forall f \in C_c(\mathbb{R}) \subseteq \mathrm{dom}\, M \subsetneq L^2(\mathbb{R})$. Thus, $(A,B) \sim_W \bigoplus_{j\in J} (M,D)$.

Useful: $(R,S) \longrightarrow (A,B)$

Heisenberg rep. of $\mathbb{R}$ $\longleftarrow$ + supplementary conditions. Heis. pair.

Theorem (Huang, 2017)
If (A,B) is a Heis. pair on H such that $\exists \mathcal{D}$ a dense $\{A,B\}$-analytic subspace of H s.t.

$$[A,B]\big|_{\mathcal{D}} = iI\big|_{\mathcal{D}},$$

then $\left(\{e^{ixB}\}_{x\in\mathbb{R}}, \{e^{i\gamma A}\}_{\gamma\in\mathbb{R}}\right)$ is a Heisenberg rep'n of $\mathbb{R}$.

Generalize to Hilbert C^*-modules...

② Covariant von Neumann Uniqueness Theorem

Theorem: Let (X, r, s) be a Heisenberg rep'n of G. Then (X,r,s) is unitarily equivalent to a direct sum of copies of $(L^2(G,A), u, v)$.

Covariant Theorem: Every (G,A,α)-Heisenberg rep. (X,ρ,r,s) is unitarily equivalent to a direct sum of copies of $(L^2(G,A,\alpha), M, u, v)$, where

- X is full (right) Hilbert A-module
- $\rho: A \longrightarrow \mathcal{L}(X)$ nondeg. $*$-rep'n
- (ρ, r) covariant rep'n of (A, G, α)
- (ρ, s) commuting rep'n's
- (r,s) satisfy WCR

- $L^2(G,A,\alpha)$: is completion of $C_c(G,A)$ as right Hilbert A-module w/ action $(f\cdot a)(x) = f(x)\alpha_x(a)$ and $\langle f, g\rangle_A = \int_G \alpha_{x^{-1}}(f(x)^* g(x))\,d\mu(x)$.
- $M: A \to \mathcal{L}(L^2(G,A,\alpha))$ is $(M(a)f)(x) = a f(x)$
- $(u_y f)(x) = \alpha_y(f(y^{-1}x))$; $(v_\gamma f)(x) = \gamma(x) f(x)$

We actually have only achieved the case when $\mathcal{A}=K(H)$. The reason why will become clear...

③ Proof of the Classical Version

① $C_0(G)\rtimes_{lt} G \cong K(L^2(G))$ Stone-von Neumann Theorem.
$\uparrow$ $\xi\rtimes U$, $\xi: C_0(G)\to B(L^2(G))$ mult., $U: G\to \mathcal{U}(L^2(G))$ l.r.r.
$\hookrightarrow$ Representations of $K(L^2(G))$ are nice

② Given a Heisenberg rep.'n (H,R,S) for G, build a nondeg. covariant $*$-representation (π_S, R) for $(C_0(G), G, lt)$ on H:
$\forall f\in C_c(\hat{G})$, $\pi_S(\hat{f}) := \int_{\hat{G}} f(\gamma) S_\gamma \, d\hat{\mu}(\gamma)$.
($\hat{f} \in C_0(G)$)

③ $(\pi_S \rtimes R)\circ(\xi\rtimes U)^{-1} : K(L^2(G)) \longrightarrow B(H)$ is a nondeg. $*$-rep'n.
so it's unitarily equivalent to $\bigoplus_{j\in J} id: K(L^2(G))\hookrightarrow B(L^2(G))$

$$\exists W: \bigoplus_{j\in J} L^2(G) \longrightarrow H \text{ unitary such that}$$

$$(\pi_S\rtimes R)\circ(\xi\rtimes U)^{-1} = ad_W\circ\left(\bigoplus_{j\in J} id\right)$$

$$\Longrightarrow \pi_S\rtimes R = ad_W\circ\left(\bigoplus_{j\in J}\xi\rtimes U\right) = ad_W\circ\left(\bigoplus_{j\in J}\pi_V\rtimes U\right).$$

④ Proof of our Covariant Version

To generalize, we had to make the following adjustments:

⓪ In defining an $(\mathcal{A},G,\alpha)$-Heisenberg rep'n and the $(\mathcal{A},G,\alpha)$-Schrödinger rep'n, we needed the latter to be one of the former, and we needed a canonical Hilbert $\mathcal{A}$-module to encode $(\mathcal{A},G,\alpha) \rightsquigarrow L^2(G,\mathcal{A},\alpha)$.

① $\underbrace{C_0(G,A)\rtimes_{lt\otimes\alpha} G}_{\sigma} \cong K(L^2(G,A,\alpha))$

$\Xi \rtimes u$ where $\Xi : C_0(G,A) \to \mathcal{L}(L^2(G,A,\alpha))$ is just $[\Xi(g)f] = gf \ \forall f \in C_c(G,A)$.

<u>Green's Imprimitivity Theorem:</u> $C_c(G,A)$ is a $C_c(G\times G, A) - A$ pre-imprimitivity bimodule w/

$$(b\cdot f)(x) = \int_G b(x,y)\,\sigma\,(f(y))\,d\mu(y)$$

$$_B\langle f,g\rangle(x,y) = f(y)^*\,\sigma_y\,(g(x))$$

$\rightsquigarrow$ $L^2(G,A,\alpha)$ is a $C_0(G,A)\rtimes_\sigma G - A$ imprimitivity bimodule. The isomorphism $\Xi\rtimes u$ is implemented by the left action of $C_0(G,A)\rtimes_\sigma A$ on $L^2(G,A,\alpha)$ as a right Hilbert A-module.

<u>Theorem</u> (Huang-I., 2018): If X and Y are Hilbert $K(H)$-modules, $\tilde{\pi}: K(X) \to \mathcal{L}(Y)$ a nondeg. $*$-rep., then $\tilde{\pi}$ is unitarily equivalent to a direct sum of copies of $'\mathrm{id}_{K(X)\to\mathcal{L}(X)}$.

<u>Proof</u> (Idea): Fix a rank-one projection $p \in K(H)$. Then

$$\underbrace{K(X\cdot p)}_{\text{Hilbert space}} \xrightarrow[\text{Bakić-Guljaš}]{\psi_X^{-1}} K(X) \xrightarrow{\tilde{\pi}} \mathcal{L}(Y) \xrightarrow[\text{Bakić-Guljaš}]{\psi_X} \mathcal{L}(Y\cdot p)$$

is a nondeg. $*$-rep'n. of $K(X\cdot p)$ into $\mathcal{L}(Y\cdot p)$. Apply Arveson's proof of unitary equivalence in this setting.

(2) Given a $(\mathcal{A}, G, \alpha)$-Heisenberg rep'n (X, p, r, s), we can define a nondeg $*$-rep'n $\Pi_{p,s} : C_0(G, \mathcal{A}) \longrightarrow \mathcal{L}(X)$ by:

$$\forall f \in C_c(\hat{G}, \mathcal{A}), \quad \Pi_{p,s}(\underset{\substack{\cap \\ C_0(G,\mathcal{A})}}{\hat{f}}) = \int_{\hat{G}} p(f(\gamma))\, s_\gamma \, d\hat{\mu}(\gamma)$$

Using $C^*(\hat{G}) \otimes \mathcal{A} \cong C_0(G, \mathcal{A})$, $\Pi_{p,s}$ extends, and $(\Pi_{p,s}, r)$ is a covariant homomorphism for $(C_0(G, \mathcal{A}), G, \sigma)$.

Let $\mathcal{A} = K(H)$.

(3) $(\Pi_{p,s} \rtimes r) \circ (\Xi \rtimes u)^{-1} : K(\underbrace{L^2(G, K(H), \alpha)) \longrightarrow \mathcal{L}(X)}_{\text{Hilbert } K(H)\text{-modules}}$ is unitarily equivalent to $\bigoplus_{j \in J} \mathrm{id} : K(L^2(G, K(H), \alpha)) \hookrightarrow \mathcal{L}(L^2(G, K(H), \alpha))$, so $\exists$ $w : \bigoplus_{j \in J} L^2(G, K(H), \alpha) \longrightarrow X$ unitary such that

$$\Pi_{p,s} \rtimes r = \mathrm{ad}_w \circ \Big(\bigoplus_{j \in J} \Xi \rtimes u\Big)$$
$$= \mathrm{ad}_w \circ \Big(\bigoplus_{j \in J} \Pi_{M,v} \rtimes u\Big)$$

"Untwist."

(5) Conclusions

- Leonard and I were able to generalize his result and then appeal to our covariant vN Uniqueness Theorem to classify pairs of (unbounded) self-adjoint operators on Hilbert $K(H)$-modules.
- We don't think we can further generalize $\mathcal{A}$ - we have to know a lot about its rep'n theory.
- Neat idea: $(K(H), G, \alpha) \rightsquigarrow$ projective representation of G as $\{U_x\}_{x \in G}$ on H where $\alpha_x(a) = U_x a U_x^{-1}$ $\forall a \in K(H)$.

We'd like to explore this perspective of classifying proj. unitary rep'ns of G.