The Covariant Stone-von Neumann Theorem

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A Historical Problem

Goal: Classify pairs (A, B) of (possibly unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$ that satisfy:

- A and B share a common dense domain $K \subseteq \mathcal{H}$ and
- $[A, B]h = ih$ for all $h \in K$. (Heisenberg Commutation Relation)

Example (Schrödinger Pair)

$Q = M_x$ and $P = -i\frac{d}{dx}$ on $L^2(\mathbb{R})$ is such a pair...is this "all" of them?

No.

Strategies for Classification

Goal: Classify pairs (A, B) of (possibly unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$ that satisfy:

- A and B share a common dense domain $K \subseteq \mathcal{H}$ and
- $[A, B]h = ih$ for all $h \in K$. (Heisenberg Commutation Relation)

Strategies:

- Make more restrictive requirements on dense subspace K.
- Classify unitary groups generated by A and B which satisfy the

  Weyl Commutation Relation.

The Weyl Commutation Relation

Definition

Let G be a l.c.a. group. A pair (R, S) of unitary representations

$$R : G \to \mathcal{U}(\mathcal{H}) \text{ and } S : \widehat{G} \to \mathcal{U}(\mathcal{H})$$

satisfy the Weyl Commutation Relation if

$$S_\gamma R_x = \gamma(x) R_x S_\gamma \qquad \forall x \in G, \gamma \in \widehat{G}.$$

Example (Schrödinger Representation)

The representations $U : G \to \mathcal{U}(L^2(G))$ and $V : \widehat{G} \to \mathcal{U}(L^2(G))$ given by

$$(V_\gamma f)(y) = \gamma(y) f(y) \text{ and } (U_x f)(y) = f(x^{-1}y); \;\; \forall y \in G$$

satisfy the Weyl Commutation Relation.

Question: Is this "all" pairs which satisfy the WCR? Yes.

Classical Stone-von Neumann Theorem

Theorem (Stone-von Neumann / von Neumann Uniqueness)

Any pair of unitary representations (S, R) of G satisfying the WCR must be $(S, R) \sim \oplus(V, U)$.

1. **Corollary.** If A and B generate unitary groups S and R which satisfy the WCR, then

$$(A, B) \sim \oplus(Q, P).$$

2. Unfortunately,

$$(A, B) \text{ is a Heisenberg pair} \not\Rightarrow (S, R) \text{ is a Heisenberg representation.}$$

3. **Moral:** Must determine *when* (A, B) generate unitary groups which satisfy the WCR...which depends on the common domain $K \subset \mathcal{H}$.

Theorem (Stone-von Neumann / von Neumann Uniqueness)

Any pair of unitary representations (S, R) *of* G *on* $\mathcal{H}$ *satisfying the WCR must be* $(S, R) \sim \oplus(V, U)$*.*

Initial Goal: Extend Stone-von Neumann Theorem to unitary group representations on Hilbert C^*-modules, denoted X.

- Define appropriate extensions of
  - "satisfying the WCR" on X, called a Heisenberg representation, and
  - the Schrödinger representation (U, V) on $L^2(G)$
- But wait!
  - Now there's a C^*-algebra $\mathcal{A}$ with right action on X,
  - and we also have a l.c.a. group G...

Larger Goal: Extend Stone-von Neumann Theorem to representations of C^*-dynamical systems on Hilbert C^*-modules.

Theorem (Huang-I., 2018)

Every $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation is unitarily equivalent to a direct sum of copies of the $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Schrödinger representation.

$(G, \mathcal{A}, \alpha)$-Heisenberg Representations

Definition

A $(G, \mathcal{A}, \alpha)$-Heisenberg module representation is a quadruple (X, ρ, r, s) with the following properties:

- X is a (full) Hilbert $\mathcal{A}$-module.
- $\rho : \mathcal{A} \to \mathcal{L}(\mathrm{X})$ is a (nondegenerate) $*$-representation.
- $r : G \to \mathcal{U}(\mathrm{X})$ and $s : \widehat{G} \to \mathcal{U}(\mathrm{X})$ are unitary group representations.
- (ρ, r) is a covariant homomorphism of $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\mathrm{X})$.
- (ρ, s) is covariant homomorphism of $(\widehat{G}, \mathcal{A}, \iota)$ into $\mathcal{L}(\mathrm{X})$.
- $s_\gamma r_x = \gamma(x) r_x s_\gamma$ for all $x \in G$ and $\gamma \in \widehat{G}$.

When $\mathcal{A} = \mathbb{C}$, this is a classical Heisenberg representation.
Warning: X may not have orthogonally complemented $\mathcal{A}$-submodules.

The $(G, \mathcal{A}, \alpha)$-Schrödinger Representation

Definition

Let $\mathrm{L}^2(G, \mathcal{A}, \alpha)$ be the completion of $C_c(G, \mathcal{A})$ as a right Hilbert $\mathcal{A}$-module with twisted action $[f \bullet a](x) := f(x)\alpha_x(a)$ for all $x \in G$ and

$$\langle f \,|\, g \rangle := \int_G \alpha_{x^{-1}} \left(f(x)^* g(x) \right) \, d\mu(x).$$

For $\phi \in C_c(G, \mathcal{A})$, define

- $\mathrm{M} : \mathcal{A} \to \mathcal{L}(\mathrm{L}^2(G, \mathcal{A}, \alpha))$ by $[\mathrm{M}(a)\phi](x) := a\phi(x)$ for all $x \in G$,
- $u : G \to \mathcal{U}(\mathrm{L}^2(G, \mathcal{A}, \alpha))$ by $[u_x \phi](y) := \alpha_x(\phi(x^{-1}y))$ for all $y \in G$,
- $v : \widehat{G} \to \mathcal{U}(\mathrm{L}^2(G, \mathcal{A}, \alpha))$ by $[v_\gamma \phi](y) := \gamma(y)\phi(y)$ for all $y \in G$.

The $(G, \mathcal{A}, \alpha)$-Schrödinger representation is $(\mathrm{L}^2(G, \mathcal{A}, \alpha), \mathrm{M}, u, v)$. When $\mathcal{A} = \mathbb{C}$, we recover the classical Schrödinger representation (U, V).

First Ingredient of Covariant Stone-von Neumann Theorem

Proposition (Huang-I., 2018)

$(\mathsf{L}^2(G,\mathcal{A},\alpha),\mathsf{M},u,v)$ *is a* $(G,\mathcal{A},\alpha)$*-Heisenberg representation.*

The first ingredient in proving Classical Stone-von Neumann Theorem is

$$C_o(G)\rtimes_{\mathrm{lt}} G \overset{\cong}{\to} \mathcal{K}\big(L^2(G)\big) \rightsquigarrow C_o(G,\mathcal{A})\rtimes_{\mathrm{lt}\otimes\alpha} G \overset{\cong}{\to} \mathcal{K}\big(\mathsf{L}^2(G,\mathcal{A},\alpha)\big)$$

Classical isomorphism is given by

$$\xi\rtimes U : C_o(G)\rtimes_{\mathrm{lt}} G \overset{\cong}{\to} \mathcal{K}\big(L^2(G)\big).$$

Replace

$$\xi : C_c(G)\to B(L^2(G)) \text{ with } \Xi : C_c(G,\mathcal{A})\to \mathcal{L}(\mathsf{L}^2(G,\mathcal{A},\alpha)).$$

Proposition (Huang-I., 2018)

$$\Xi\rtimes u : C_o(G,\mathcal{A})\rtimes_{\mathrm{lt}\otimes\alpha} G \overset{\cong}{\to} \mathcal{K}\big(\mathsf{L}^2(G,\mathcal{A},\alpha)\big)$$

Main Ideas of Isomorphism

Proposition (Huang-I., 2018)

$$\Xi \rtimes u : C_o(G, \mathcal{A}) \rtimes_{\mathrm{lt}\otimes\alpha} G \xrightarrow{\cong} \mathcal{K}\big(\mathrm{L}^2(G, \mathcal{A}, \alpha)\big)$$

- The Hilbert $\mathcal{A}$-module $\mathrm{L}^2(G, \mathcal{A}, \alpha)$ is a $C_o(G, \mathcal{A}) \rtimes_{\mathrm{lt}\otimes\alpha} G - A$ imprimitivity bimodule by Green's Imprimitivity Theorem.
- In general, if X is a $B - A$ imprimitivity bimodule, then $B \cong \mathcal{K}(\mathrm{X}_{\mathcal{A}})$. .
- We verified this canonical isomorphism is implemented by $\Xi \rtimes u$.

Second Ingredient

Given (S, R) on $\mathcal{H}$, one can use $\mathcal{F} : C_c(\widehat{G}) \to C_o(G)$ to build

- $\pi_S : C_o(G) \to \mathcal{B}(\mathcal{H})$
- $\rightsquigarrow (\pi_S, R)$ is a covariant representation of $(G, C_o(G), \mathrm{lt})$ on $\mathcal{H}$

Proposition

Given (X, ρ, r, s), can construct a covariant representation $(\Pi_{\rho,s}, r)$ of $(C_o(G, \mathcal{A}), G, \mathrm{lt} \otimes \alpha)$ on X.

Classically:

$$\mathcal{K}(L^2(G)) \overset{(M \rtimes U)^{-1}}{\longrightarrow} C_o(G) \rtimes_{\mathrm{lt}} G \overset{\pi_S \rtimes R}{\longrightarrow} \mathcal{B}(\mathcal{H})$$

Covariant:

$$\mathcal{K}(\mathrm{L}^2(G, \mathcal{A}, \alpha)) \overset{(\Xi \rtimes u)^{-1}}{\longrightarrow} C_o(G, \mathcal{A}) \rtimes_{\mathrm{lt} \otimes \alpha} G \overset{\Pi_{\rho,s} \rtimes r}{\longrightarrow} \mathcal{L}(\mathsf{X})$$

An Observation of Second Ingredient

Classically:

$$\mathcal{K}(L^2(G)) \xrightarrow{(M \rtimes U)^{-1}} C_o(G) \rtimes_{\mathrm{lt}} G \xrightarrow{\pi_S \rtimes R} \mathcal{B}(\mathcal{H})$$

This is a nondegenerate $*$-representation of $\mathcal{K}(L^2(G)) \Rightarrow$ unitarily equivalent to a direct sum of copies of the identity representation.

Covariant:

$$\mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha)) \xrightarrow{(\Xi \rtimes u)^{-1}} C_o(G, \mathcal{A}) \rtimes_{\mathrm{lt} \otimes \alpha} G \xrightarrow{\Pi_{\rho,s} \rtimes r} \mathcal{L}(\mathsf{X})$$

Theorem (Huang-I., 2018)

Let Y be a Hilbert $\mathcal{K}(\mathcal{H})$-module. Any (nondegenerate) $*$-representation $\mathcal{K}(\mathsf{Y}) \to \mathcal{L}(\mathsf{X})$ is unitarily equivalent to a direct sum of the identity representation.

Restricting $\mathcal{A}$ to $\mathcal{K}(\mathcal{H})$

Theorem (Huang-I., 2018)

Let Y be a Hilbert $\mathcal{K}(\mathcal{H})$-module. Any (nondegenerate) $*$-representation $\mathcal{K}(\mathsf{Y}) \to \mathcal{L}(\mathsf{X})$ is unitarily equivalent to a direct sum of the identity representation.

Corollary

$\mathcal{K}(\mathsf{L}^2(G, \mathcal{K}(\mathcal{H}), \alpha)) \xrightarrow{(\Xi \rtimes u)^{-1}} C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_{\mathrm{lt} \otimes \alpha} G \xrightarrow{\Pi_{\rho,s} \rtimes r} \mathcal{L}(\mathsf{X})$ *is unitarily equivalent to a direct sum of the identity representation.*

Main Theorem

Theorem (Huang-I., 2018)

Every $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation is unitarily equivalent to a direct sum of copies of the $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Schrödinger representation.

The final observations to make:

- $\mathcal{K}(\mathrm{L}^2(G, \mathcal{K}(\mathcal{H}), \alpha)) \overset{(\Xi \rtimes u)^{-1}}{\longrightarrow} C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_{\mathrm{lt} \otimes \alpha} G \overset{\Pi_{\rho,s} \rtimes r}{\longrightarrow} \mathcal{L}(\mathrm{X}) \sim$ direct sum of copies of the identity representation
- $\Xi = \Pi_{\mathrm{M},v}$, so $\Xi \rtimes u = \Pi_{\mathrm{M},v} \rtimes u$
- Therefore, $\Pi_{\rho,s} \rtimes r$ is unitarily equivalent to a direct sum of copies of $\Pi_{\mathrm{M},v} \rtimes u$
- Untwist to get $\rho \sim \bigoplus \mathrm{M}$, $s \sim \bigoplus v$, and $r \sim \bigoplus u$.

Thank you!