Analytic Vectors of a Weakly-Defined Derivation

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Ample analytic vectors corresponds, in some sense, to stronger continuity.

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Goal: Use the continuity of a weakly-defined derivation δ_D on $\mathcal{B}(\mathcal{H})$ to examine its analytic vectors $\mathcal{A}_\delta$.

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1. Define a second notion of analyticity with respect to a flow on a Banach space.
2. Relate this notion and the original definition.
3. Give a constructive proof of analytic vector density in the SOT.

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Fix a closed subspace $F \subseteq X^*$. Let $\{\sigma_t\}_{t\in\mathbb{R}}$ be a $\sigma(X, F)$-continuous group of isometries on X.

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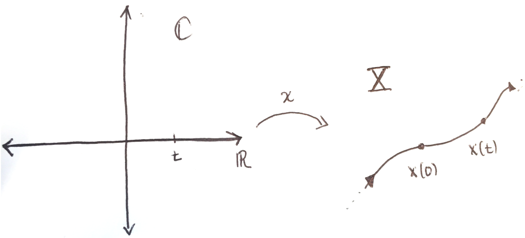

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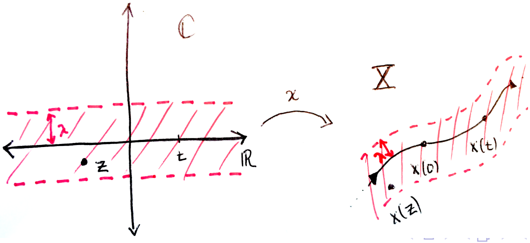

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Example

Let $\{U_t\}_{t\in\mathbb{R}}$ be a strongly-continuous group of unitaries on $\mathcal{H}$. Given $h \in \mathcal{H}$, define $h(t) := U_t h$. Then h is analytic for $\{U_t\}_{t\in\mathbb{R}}$ if $\forall\ k \in \mathcal{H}$,

$$t \mapsto \langle h(t), k\rangle \text{ extends to an analytic function on a complex region } I_\lambda.$$

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If $h \in \mathcal{H}$, then

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Proposition

If $h \in \mathcal{H}$, then

$$h \in \mathcal{A}_D \iff h \text{ is analytic for } \{e^{itD}\}_{t\in\mathbb{R}}.$$

In fact, if S is the infinitesimal generator of a $\sigma(X, F)$-continuous group of isometries $\{\sigma_t\}_{t\in\mathbb{R}}$ on X, then

$$x \in \mathcal{A}_S \iff x \text{ is analytic for } \{\sigma_t\}_{t\in\mathbb{R}}.$$

Exploiting this relation for δ_D

For each $t \in \mathbb{R}$, define $\alpha_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\alpha_t(x) = e^{itD} x e^{-itD} \quad \forall x \in \mathcal{B}(\mathcal{H}).$$

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An operator $x \in \mathcal{B}(\mathcal{H})$ is weakly D-differentiable if $\exists y \in \mathcal{B}(\mathcal{H})$ s.t.

$$\lim_{t\to 0} \left| \left\langle \left(\frac{\alpha_t(x) - x}{t} - y \right) h, k \right\rangle \right| = 0 \text{ for all } h, k \in \mathcal{H}.$$

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Notation: $x \in \mathrm{Dom}\,(\delta)$ and $\delta(x) = -iy$, so $\delta : \mathrm{Dom}\,(\delta) \to \mathcal{B}(\mathcal{H})$.

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- $t \mapsto \left\langle e^{itD} h, g \right\rangle$ and $t \mapsto \left\langle f, e^{itD} k \right\rangle$ are n-times differentiable.
- thus, the product $t \mapsto \left\langle e^{itD} h, g \right\rangle \left\langle f, e^{itD} k \right\rangle$ is n-times differentiable.

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- BUT the extension of $t \mapsto \left\langle e^{itD} k, f \right\rangle$ is conjugate to the extension of $t \mapsto \left\langle f, e^{itD} k \right\rangle$...

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- Instead of taking $h, k \in \mathcal{A}_D$ and trying to show $h \otimes k^* \in \mathcal{A}_\delta$,

- Choose φ_k to be analytic for $\mathcal{R}D\mathcal{R}^{-1}$, and show $h \otimes k^* \in \mathcal{A}_\delta$.

Thank you!