OTRS
Spring 2018

Date	Topic	Speaker
2/27, 3/6	Ch. 2 of Lance (p.14-20) "Multipliers & morphisms"	Catherine Godfrey
3/13	Ch. 3 of Lance (p. 21-25) "Projections & unitaries" part 1	Sean Gravelle
3/27	Ch. 3 of Lance (p. 26-30) "Projections & unitaries" part 2	Sean Gravelle
4/3	Tensor products of C^*-alg.'s. <u>Lots of examples</u>	Stephanie Prahl
3/20	Spring Break	Catherine Godfrey
4/10	Ch. 4 of Lance (p. 31-38) "Tensor Products" Part 1 of A-modules	Juliana Bukoski
4/17	Ch. 4 of Lance (p. 39-44) "Tensor products" Part 2 of A-modules.	Juliana Bukoski -OR- Lara Ismert.
4/24	Intro to <u>Quantum</u> Groups	Robert Huben
	More on Quantum Groups / Applications	

Hilbert C^*-modules

Prerequisites: vector bundles, C^*-algebras

Applications: KK theory, Morita equivalence, groupoid C^*-algebras, C^*-correspondences

Resources: Lance - "Hilbert C^*-modules, a toolkit for Operator Algebraists"

Raeburn, Williams -

Quantum Groups

Prerequisites: Harmonic analysis, KMS Weights, algebra :)

Applications: Crossed Products, Hopf algebras, TQFT

Resources:

$\boxed{\text{Hilbert } C^*\text{-Modules}}$

2/4/18

<u>Give Outline</u>: continuous H. bundle $\rightsquigarrow$ H. $C(X)$-modules $\rightsquigarrow$ H. A-modules
- example in D.G. — motivates — motivates

<u>Sources</u>: Weaver "Mathematical Quantization" Ch. 9, Lance Ch. 1.

(1) Hilbert Bundles

<u>Defn</u>: Let X be a compact Hausdorff space.

- A <u>covering space</u> of X is a topological space (Y, p) and a continuous open surjection $p: Y \to X$.

- A <u>continuous Hilbert bundle</u> over X is a covering space $\mathcal{H}$ such that $H_x := p^{-1}(x)$ is a Hilbert space for each $x \in X$ such that

  (a) $y \mapsto \|y\|$ is continuous from $\mathcal{H}$ to $\mathbb{R}$

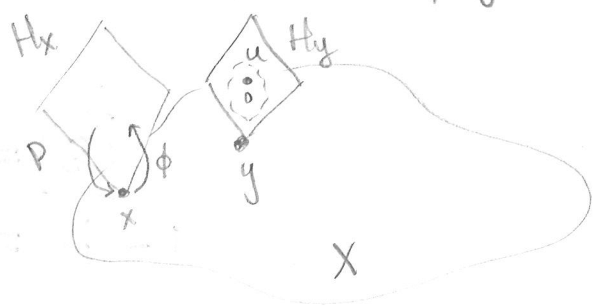
  (b) $(y, z) \mapsto y + z$ is cts. from $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$

  (c) $y \mapsto ay$ is cts. on $\mathcal{H}$ for each $a \in \mathbb{C}$

  (d) for any neighborhood $\mathcal{U}$ of $0 \in H_x$, $\exists\, \mathcal{U}'$ of $x \in X$ and $\varepsilon > 0$ s.t.

  nbhd

  $$\{ y \in \mathcal{H} : p(y) \in \mathcal{U}', \|y\| < \varepsilon \} \subset \mathcal{U}.$$



$\mathcal{H} = \bigcup_{x \in X} H_x$, with a topology w.r.t. which it is a topological vector space, the norm is cts, and has a nbhd base defined interms of norm balls and preimages of p.

- <u>A section of $\mathcal{H}$</u> is a function $\varphi: X \to \mathcal{H}$ such that $\varphi(x) \in H_x$ $\forall x \in X$. The set of all continuous sections of $\mathcal{H}$ is denoted $S(\mathcal{H})$. (They're all cts.)

<u>Note</u>: $\varphi \in S(X)$ if whenever $\tilde{\mathcal{U}} \subset \bigsqcup_{x \in X} H_x$ is open, $\varphi^{-1}(\tilde{\mathcal{U}})$ is open in X. Note that $\varphi^{-1}(\tilde{\mathcal{U}}) \overset{?}{=} p(\tilde{\mathcal{U}})$, which is an open map.

Examples

(a) X smooth manifold, at each $x \in X$ $\exists\, T_x$, tangent (over $\mathbb{R}$) vector space, and $TX := \bigsqcup_{x \in X} T_x$ is called the tangent bundle. X is Riemannian if T_x has inner product $\langle \cdot, \cdot \rangle_x$ which varies "smoothly" w.r.t. x. By complexifying TX, TX becomes a continuous Hilbert bundle.

Check: Define $p : TX \to X$ by $T_x \ni v_x \xmapsto{p} x \in X$.

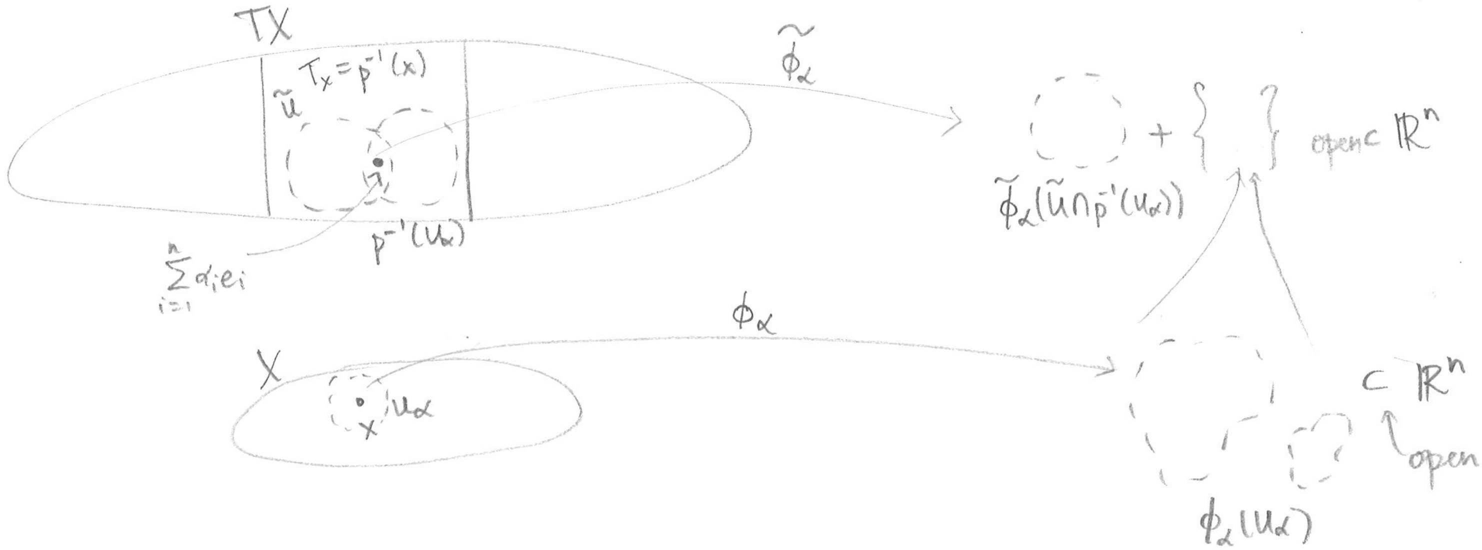
Topology on TX: Suppose X is an n-dim'l smooth mfld w/ smooth atlas of charts $\{\underbrace{U_\alpha}_{\text{open cover}}, \underbrace{\phi_\alpha}_{: U_\alpha \to \mathbb{R}^n \text{ diffeo.}}\}$. This makes T_x iso. to $\mathbb{R}^n$ for each $x \in U$.

Define

$$\tilde{\phi}_\alpha : p^{-1}(U_\alpha) \to \mathbb{R}^{2n} \text{ by } \tilde{\phi}_\alpha(x, \underbrace{\sum_{i=1}^n \alpha_i e_i}_{\in T_x,\ \text{basis } \{e_i\}_{i=1}^n}) := (\phi_\alpha(x), \alpha_1, \dots, \alpha_n).$$

We say $\tilde{U} \subset TX$ is open iff $\tilde{\phi}_\alpha(\tilde{U} \cap p^{-1}(U_\alpha))$ is open in $\mathbb{R}^{2n}$ $\forall \alpha$.

The atlas of charts $\{\tilde{\phi}_\alpha, \tilde{U}_\alpha\}$ makes TX into a smooth mfld.



$\rightsquigarrow$ Equip each T_x w/ inner product & complexify it.

Back to Sections

Let $C(X) = \{f : X \to \mathbb{C} \mid f \text{ is cts.}\}$. Then, weighting $\phi \in S(X)$ by $f \in C(X)$ is a continuous section: $x \mapsto \phi(x) f(x) \in H_x$.

Hence, $S(X)$ is a $C(X)$-module:

(i) Let $\varphi, \psi \in S(X)$. Then, given $\tilde{\mathcal{U}} \subset \mathcal{H}$, we know $\tilde{\phi}_\alpha(\tilde{\mathcal{U}} \cap p^{-1}(\mathcal{U}_\alpha))$ is open in $\mathbb{R}^{2n}$. Consider

$(\varphi + \psi)^{-1}(\tilde{\mathcal{U}}) \subset X$. Suppose $\tilde{\mathcal{U}}$ is an open nbhd of $0 \in H_x$ for some $x \in X$. Then, $\exists\, \mathcal{U}'$ open in X containing x and $\varepsilon > 0$ such that

$$\{v \in \mathcal{H} : p(v) \in \tilde{\mathcal{U}}, \|v\| < \varepsilon\} \subset \mathcal{U}.$$

Note that

$$(\varphi + \psi)^{-1}(\tilde{\mathcal{U}}) = \{y \in X : (\varphi + \psi)(y) \in \tilde{\mathcal{U}}\}$$
$$= p(\tilde{\mathcal{U}}) \leftarrow \text{open.}$$
$$= \{x\}.$$

(ii) Distribute prop., etc.

$= x \mapsto (\varphi(x), \psi(x)) \mapsto \langle \varphi(x), \psi(x) \rangle.$

(iii) $x \mapsto \langle \varphi(x), \psi(x) \rangle =: \langle \varphi, \psi \rangle(x)$. is a cts. fncs on X.

Defn. X cmpt. T^2 space, $\mathcal{E}$ a $C(X)$-module. A $C(X)$-valued semi inner product on $\mathcal{E}$ is a map

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to C(X)$$

such that

(a) $\langle f\phi_1 + g\phi_2, \psi \rangle = f\langle \phi_1, \psi \rangle + g\langle \phi_2, \psi \rangle$

(b) $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$

(c) $\langle \phi, \phi \rangle \geq 0$

Let $|\phi| := \langle \phi, \phi \rangle^{1/2} \in C(X)$, and define $\|\phi\| := \| \, |\phi| \, \|_\infty$.

<u>Def'n</u>: Let $\mathcal{A}$ be a C^*-algebra. An <u>inner product $\mathcal{A}$-module</u> E is a right $\mathcal{A}$-module ($\lambda(xa) = (\lambda x)a = x(\lambda a)$ $\forall x \in E$, $\forall a \in \mathcal{A}$, $\forall \lambda \in \mathbb{C}$) with a map $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}$ s.t.

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ $\forall x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$

(ii) $\langle y, x \rangle = \langle x, y \rangle^*$

(iii) $\langle x, ya \rangle = \langle x, y \rangle a$

(iv) $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0 \Rightarrow x = 0$.

For $x \in E$, define $\|x\| := \|\langle x, x \rangle\|^{1/2}$. Then $\|\langle x, y \rangle\| \leq \|x\| \, \|y\|$. We call E a Hilbert $\mathcal{A}$-module if it is complete w.r.t. this norm.

$\boxed{\text{Examples}}$

1) Any Hilbert space H is an inner product $\mathbb{C}$-module.

2) Any C^*-alg. $\mathcal{A}$ is an inner-product $\mathcal{A}$-module:

$$\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad \langle a, b \rangle := a^* b.$$

(i) $$\langle a, \alpha b + \beta c \rangle = a^*(\alpha b + \beta c) = \alpha a^* b + \beta a^* c = \alpha \langle a, b \rangle + \beta \langle a, c \rangle$$

(ii) $\langle b, a \rangle = b^* a = (a^* b)^* = \langle a, b \rangle^*$ Show C.S. inequal?

(iii) $\langle a, bc \rangle = a^* b c = (a^* b) c = \langle a, b \rangle c$

(iv) $\langle a, a \rangle = a^* a \geq 0$; $a^* a = 0 \Rightarrow a = 0$

3) $\{E_i\}_{i=1}^n$ a family of $\mathcal{A}$-modules, $\bigoplus_{i=1}^n E_i$ is an $\mathcal{A}$-module with $\langle x, y \rangle := \sum_{i=1}^n \langle x_i, y_i \rangle$; $x = (x_i)$, $y = (y_i)$.

$\{E_i\}_{i \in I}$ ", then $\bigoplus E_i := \{(x_i) : \sum_{i \in I} \langle x_i, x_i \rangle \text{ converges}\}$. Define

$$\langle (x_i), (y_i) \rangle := \sum_{i \in I} \langle x_i, y_i \rangle.$$

4) H a Hilbert space, $\mathcal{A}$ a C^*-alg, then $H \otimes_{alg} \mathcal{A}$ is right $\mathcal{A}$-module and

$$\langle \cdot,\cdot \rangle : (H \otimes_{alg} \mathcal{A}) \times (H \otimes_{alg} \mathcal{A}) \longrightarrow \mathcal{A} \qquad \langle h\otimes a, k\otimes b\rangle := \langle h,k\rangle a^* b$$

extend to finite sums.

- Exercise in OTRS -

Pos.-def.: Let $\sum_{i=1}^n h_i \otimes a_i \in H \otimes_{alg} \mathcal{A}$. Then, if $K = \operatorname{span}\{h_1,\dots,h_n\}$, $\exists$ o.n.b. $\{e_j\}_{j=1}^m$ such that $h_i = \sum_{j=1}^m \lambda_{ij} e_j$ $\forall i=1,\dots,n$. Now,

$$\left\langle \sum_{i=1}^n h_i \otimes a_i , \sum_{k=1}^n h_k \otimes a_k \right\rangle = \quad \dots$$

5) (Localization) $\mathcal{A}$ unital C^*-algebra, $\mathcal{B} \subset \mathcal{A}$ C^*-subalgebra containing the identity of $\mathcal{A}$.

Defn: A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ which is a contractive idempotent is called a conditional expectation. (Kind of like a retract)

- Φ is always positive
- $\Phi(bac) = b\,\Phi(a)\,c \quad \forall a \in \mathcal{A},\ \forall b,c \in \mathcal{B}$

It is faithful if whenever $0 \le a \in \mathcal{A}$, $\Phi(a) = 0 \Rightarrow a = 0$.

Let E be an inner product $\mathcal{A}$-module. Then

$$\langle x,y\rangle_{\mathcal{B}} := \Phi(\langle x,y\rangle_{\mathcal{A}}) \quad \text{for } x,y \in E$$

defines an inner product on $\mathcal{B}$ and makes E into an inner product $\mathcal{B}$-module.

//

Properties of Hilbert C^*-modules

- $\|x\|_E = \sup\{\|\langle x,y\rangle\| : y\in E, \|y\|\le 1\}$

- $\langle y,x\rangle\langle x,y\rangle \le \|\langle x,x\rangle\| \langle y,y\rangle$ $\forall x,y\in E$

  in ordering given by positivity

<u>Proof</u>: Sps. $\|\langle x,x\rangle\|=1$ WLOG.

For $a\in\mathcal{A}$,

$$
\begin{aligned}
0 &\le \langle xa-y, xa-y\rangle \\
&= \langle xa,xa\rangle - \langle xa,y\rangle - \langle y,xa\rangle + \langle y,y\rangle \\
&= \langle xa,x\rangle a - \langle y,xa\rangle^* - \langle y,x\rangle a + \langle y,y\rangle \\
&= \langle x,xa\rangle^* a - (\langle y,x\rangle a)^* - \langle y,x\rangle a + \langle y,y\rangle \\
&= a^*\langle x,x\rangle a - a^*\langle x,y\rangle - \langle y,x\rangle a + \langle y,y\rangle \\
&\le a^*a - a^*\langle x,y\rangle - \langle y,x\rangle a + \langle y,y\rangle
\end{aligned}
$$

If $\langle x,x\rangle \ge 0$, then $a^*\langle x,x\rangle a \le \|\langle x,x\rangle\| a^*a$.

For $a=\langle x,y\rangle$ we've done. ▨

- For $F\subseteq E$ a closed $\mathcal{A}$-submodule, define $F^\perp := \{y\in E : \langle x,y\rangle = 0\ \forall x\in F\}$

  Then
  - $F^\perp$ is a closed submodule of E
  - $F\oplus F^\perp \subseteq E$
  - $F\subseteq (F^\perp)^\perp$

Properties of Hilbert spaces

- $\|h\| = \sup\{|\langle h,k\rangle| : k\in H, \|k\|\le 1\}$.

- $|\langle h,k\rangle| \le \|h\|\,\|k\|$ $\forall h,k\in H$

  (Cauchy-Schwarz)

- For $K\subseteq H$ a closed subspace, define $K^\perp := \{k\in H : \langle h,k\rangle = 0\ \forall h\in K\}$.

  Then
  - $K^\perp$ is a closed subspace of H
  - $H = K\oplus K^\perp$
  - $K = (K^\perp)^\perp$.

Ex: $\mathcal{A}=C(X)$, $E=\mathcal{A}$, $Y\subset X$ closed, $\overline{Y^c}=X$. Then $F:=\{f\in\mathcal{A} : f(Y)=0\}$ has $F^\perp = 0$.

<u>Moral</u>: Hilbert spaces are a good intuition for Hilbert C^*-modules, but sometimes extra conditions are required for nice properties to hold.

$\boxed{\text{More analogies}}$ Let E, F be Hilbert $\mathcal{A}$-modules, H, K Hilbert spaces.

<table>
<tr><th>$B_{\mathcal{A}}(E,F)$</th><th>$B(H,K)$</th></tr>
<tr><td>• <u>Defn</u>: $t \in B_{\mathcal{A}}(E,F)$ if $t : E \to F$ and $\exists\, t^* : F \to E$ such that $\langle tx, y\rangle = \langle x, t^*y\rangle \quad \forall x \in E$ and $y \in F$
Every element is $\mathcal{A}$ linear and $\mathcal{A}$-bounded:
<u>Pf</u>: (1) t is $\mathcal{A}$-linear.
$\langle$
(2) <u>bdd</u>: For each $x \in E_1 \subset E$, define $f_x : F \to \mathcal{A}$ by $f_x(y) := \langle tx, y\rangle$. Then $\|f_x(y)\| = \|\langle tx, y\rangle\| = \|\langle x, t^*y\rangle\| \leq \|t^*y\|$, $\forall x \in E_1$. Thus, the family $\{f_x : x \in E_1\}$ is bounded by B-S, so $\sup_{\|x\|,\|y\|\leq 1} \|\langle tx, y\rangle\| \leq \sup_{x \in E_1} \|f_x\| < \infty$. //
Converse is false.</td><td>• <u>Defn</u>: $T \in B(H,K)$ if $T : H \to K$ is $\mathbb{C}$-linear and $\|T\| := \sup_{\|h\|\leq 1} \|Th\| < \infty$.
Every elt. has a unique $T^* : K \to H$ such that $\langle Th, k\rangle = \langle h, T^*k\rangle$, $\forall h \in H$, $\forall k \in K$.</td></tr>
<tr><td>• $B_{\mathcal{A}}(E)$ is a C^*-alg.
– Banach subalg. of all bdd operators on E.
– $\|t^*t\| = \sup\{\|t^*tx\| : x \in E_1\}$
$\overset{C.S.}{\geq} \sup\{\|\langle t^*tx, x\rangle\| : x \in E_1\}$
$= \sup\{\|\langle tx, tx\rangle\| : x \in E_1\}$
$= \|t\|^2$.</td><td>• $B(H)$ is a C^*-algebra.</td></tr>
</table>

• $K_A(E,F)$

Define, for $x \in E, y \in F$, $\theta_{x,y}: E \to F$ by, for $z \in E$,

$$\theta_{x,y}(z) := y\langle x, z\rangle$$

Then, for $w, z \in E$, $a \in A$,

$$\theta_{x,y}(wa+z) = y\langle x, wa+z\rangle$$
$$= y\langle x, wa\rangle + y\langle x, z\rangle$$
$$= y\langle x, w\rangle a + y\langle x, z\rangle$$
$$= \theta_{x,y}(w)a + \theta_{x,y}(z)$$

and $\forall z \in E_1$,

$$\|\theta_{x,y}(z)\| = \|y\langle x,z\rangle\|$$
$$\leq \|y\| \, \|\langle x,z\rangle\|$$
$$\leq \|y\| \cdot \|x\|$$

so $\theta_{x,y}$ is bdd. & A-linear. Moreover, for $v \in F$,

$$\langle \theta_{x,y}(z), v\rangle = \langle y\langle x,z\rangle, v\rangle$$
$$= \langle v, y\langle x, z\rangle\rangle^*$$
$$= (\langle v, y\rangle\langle x, z\rangle)^*$$
$$= \langle x,z\rangle^* \langle v,y\rangle^*$$
$$= \langle z,x\rangle\langle y,v\rangle$$
$$= \langle z, x\langle y,v\rangle\rangle$$
$$= \langle z, \theta_{yx}(v)\rangle$$

So $\theta_{xy}^* = \theta_{yx}$. $\Rightarrow \theta_{xy} \in B_A(E,F)$.

• Compacts $K(H)$

$$K(H) = \overline{F(H)} = \overline{\text{span}}\{(h \otimes k)(g) := \langle h, g\rangle k\}$$

Let $K_A(E,F) := \overline{\text{span}}\{\theta_{xy} : x \in E, y \in F\}$

- $K_A(E,F)$ is an ideal of $B_A(E,F)$
- They're not necessarily compact, though.
  - A unital, $\theta_{1,1}$ is identity on A as an A-module.

Examples:

(1) $E = A$, then $K(E) \cong A$.

Proof: Define a map

$\varphi: \{\theta_{xy} : x, y \in A\} \to A$, $\theta_{xy} \mapsto xy^*$.

Extend linearly.

Then $\operatorname{ran} \varphi \supseteq A_+$, and we know $A = \overline{\text{span}\, A_+}$. $\leftarrow$ existence of app. unit. Also, if $xy^* = ab^*$, then for $z \in E$, $w \in E$,

$$\theta_{xy}(z) = y\langle x, z\rangle = yx^*z$$
$$= ba^*z$$
$$= b\langle a, z\rangle$$
$$= \theta_{ab}(z)$$

so $\theta_{xy} = \theta_{ab}$. It's also isometric. //

(2) If A is unital, then $K_A(A) = B_A(A)$.

Topology on $B_A(E,F)$

Note that $B(H,K)$ has many topologies, includeing, but not limited to, the norm, SOT, and WOT. In contrast, we will only consider one norm on $B_A(E,F)$.

<u>Defn</u>: Consider the family of seminorms given by $x \in E, y \in F$:

$$p_x : B_A(E,F) \to \mathbb{R} \qquad p_y^* : B_A(E,F) \to \mathbb{R}$$

$$p_x(t) := \|tx\| \qquad p_y^*(t) := \|t^* y\|.$$

The topology on $B_A(E,F)$ which makes $\{p_x, p_y^* : x \in E, y \in F\}$ continuous is called the <u>strict topology</u>.

<u>Proposition</u>: $\overline{(K(E,F))_1}^{\text{strict}} = (B(E,F))_1$.

<u>Proof</u>: Let $(e_i)_{i \in I}$ be an app. unit for C^*-alg $K(E)$. We'll show $e_i x \xrightarrow[\|\cdot\|]{} x$ as $i \to \infty$ $\forall x \in E$. Then, $\forall t \in B(E,F)$, $x \in E$, $y \in F$, we'll have:

$$\|(te_i - t)x\|, \; \|(e_i t^* - t^*)y\| \to 0 \text{ as } i \to \infty$$

<u>Note</u>: $E\langle E,E\rangle$ is dense in E. So, suffices to show

$$e_i x \xrightarrow[\substack{\|\cdot\| \\ i \to \infty}]{} x \quad \forall x \in E\langle E,E\rangle.$$

Let $x = w\langle u,v\rangle$, then

$$e_i x = e_i \underbrace{\theta_{uw}}_{\in K(E)}(v) -$$

Bounded linear fnc'ls on E

Fix $x \in E$. Then $t_x y := \langle x, y \rangle$ is a map from E to A (two A-modules) which is adjointable: $t_x^* a = xa$.

Indeed, for $y \in E$, $a \in A$,

$$\langle t_x y, a \rangle = \langle \langle x, y \rangle, a \rangle$$
$$= \langle x, y \rangle^* a \quad \text{(by defn of } \langle \cdot, \cdot \rangle \text{ on } A \text{ as a module over itself)}$$
$$= \langle y, x \rangle a$$
$$= \langle y, xa \rangle$$
$$= \langle y, t_x^* a \rangle.$$

Therefore, $t_x \in B_A(E, A)$. Moreover, $t_x \in K_A(E, A)$. Indeed, whenever $x = za^*$ for some $z \in E$ and $a \in A$, we have

$$t_x y = t_{za^*} y = \langle za^*, y \rangle$$
$$= \langle y, za^* \rangle^*$$
$$= (\langle y, z \rangle a^*)^*$$
$$= a \langle y, z \rangle^*$$
$$= a \langle z, y \rangle$$
$$= \theta_{z,a}(y)$$

Fact: EA is dense in E.

Observe that $\|t_x y\| = \|\langle x, y \rangle\| \leq \|x\| \cdot \|y\|$ so $\|t_x\| \leq \|x\|$ and $\|x\|^2 = \|\langle x, x \rangle\| = \|t_x(x)\| \leq \|t_x\| \cdot \|x\|$, so $\|x\| = \|t_x\|$.

Therefore, $x \mapsto t_x$ is isometric, and so $z_n a_n^* \to x$ implies $t_{z_n a_n^*} \xrightarrow{\|\cdot\|} t_x$. Therefore, since $K_A(E, A)$ is closed, $t_x \in K_A(E, A)$

$\rightsquigarrow$ Can identify E w/ "$E^* = K_A(E, A)$."

$\rightsquigarrow$ If $E = A$, we can identify A with $K(A)$

Chapter 2: Multipliers & Morphisms

<u>Defn</u>: An ideal I of a ring R is <u>essential</u> if $rI = 0 \Rightarrow r = 0$.

If A is a (nonunital) C^*-algebra which is an essential ideal of B then B injects into the multiplier algebra $M(A)$.
(B ← unital C^*-alg.)

$$\begin{tikzcd} & M(A) \\ A \arrow[ur] \arrow[r, hook, "\text{ess. ideal}"'] & B \arrow[u, dashed, "\exists"'] \end{tikzcd}$$

<u>Commutative Analogue</u>
$A = C_0(X)$.
$Y := X \cup \{\infty\}$ gives $A \subseteq C(Y)$ minimal
$Z := \beta X$ gives $A \subseteq C(Z)$ maximally
i.e, $M(A) \cong C(Z)$

<u>Method 1</u>: Construct $M(A)$ using double centralizers.
<u>Method 2</u>: Hilbert C^*-Modules:

$$A \sim K(A) \quad \text{(as a dual space)}$$

$$M(A) \sim B(A)$$

<u>Makes sense</u>:
* $K(A)$ is an essential ideal of $B(A)$.
* If A is unital then $K(A) = B(A)$.

<u>Goal</u>: Look @ representations of A into $B_C(E)$ for some Hilbert C-module E.

<u>Defn</u>: A $*$-h'ism $\alpha: A \longrightarrow B_C(E)$ is <u>nondegenerate</u> if $\overline{\alpha(A)E} = E$.

Note:
$$\alpha(A)E = \left\{ \sum_{i=1}^{n} \alpha(a_i)x_i : a_i \in A,\ x_i \in E \right\}$$

Proposition 2.1 (Lance) $\mathcal{A}, B, C$ C^*-algebras, $\mathcal{A}$ an ideal in B, and let E be a Hilbert C-module. Then, if $\alpha : \mathcal{A} \to B_C(E)$ is a nondegenerate $*$-h'ism, α extends uniquely to a $*$-h'ism $\overline{\alpha}$ of $\mathcal{B}$ into $B_C(E)$. Moreover, if $\mathcal{A}$ is essential in $\mathcal{B}$ and α is injective, then $\overline{\alpha}$ is injective.

Proof:

1. Define $\overline{\alpha}(b)$ on the dense subset $\alpha(A)E$ by
$$\overline{\alpha}(b)\left(\sum_{i=1}^{n} a_i x_i\right) := \sum_{i=1}^{n} \alpha(\underbrace{ba_i}_{\in \mathcal{A}})\, x_i$$
Thus $\overline{\alpha}(b)$ well-defined on $\alpha(A)E$ and bounded, so this extends as a bounded map to $\overline{\alpha}(b) : \overline{\alpha(A)E} = E \to E$.

2. Moreover, $\overline{\alpha}(b^*) = \overline{\alpha}(b)^*$ so $\overline{\alpha}(b) \in B_C(E)$.

3. Now, $b \mapsto \overline{\alpha}(b)$ is a well-defined $*$-h'ism

4. If α inj., then $\ker \overline{\alpha} \cap \mathcal{A} = (0) \Rightarrow \ker \overline{\alpha} = (0)$, so $\overline{\alpha}$ is inj.

$\square$

Example: $C = E = \mathcal{A}$, $\alpha : \mathcal{A} \to B(\mathcal{A})$ by $\alpha(a) = t_a$. Hence, if $\mathcal{A} \subseteq \mathcal{B}$ as an essential ideal, then $\overline{\alpha} : \mathcal{B} \to B(\mathcal{A})$ is an injective $*$-h'ism. Hence, $B(\mathcal{A})$ has desired maximality property. Uniqueness can be shown:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, dashed, shift left, "\exists!\overline{\alpha}"] & B(\mathcal{A}) \arrow[l, dashed, shift left, "\exists!\beta"] \arrow[r, dashed, "I"] & B(\mathcal{A}) \\
& \mathcal{A} \arrow[ul, "ess."] \arrow[u, "\alpha", "ess."'] \arrow[ur, bend right, "\alpha"'] &
\end{tikzcd}$$

$\rightsquigarrow$ $\overline{\alpha}\beta = I$, by uniqueness of extensions

Theorem 2.2 (Lance) If $\mathcal{A}$ is a (possibly nonunital) C^*-alg.

(a) $B(\mathcal{A})$ is essential extension of $K(\mathcal{A})$

(b) It is unique up to isomorphism.

<u>Proposition 2.3</u> (Lance)
Let $\mathcal{A}, \mathcal{C}$ be C^*-alg. and E a Hilbert $\mathcal{C}$-module. Suppose $\alpha : \mathcal{A} \to B_{\mathcal{C}}(E)$ is a nondegenerate injective $*$-h'ism, and let B be the <u>idealiser</u> of $\alpha(\mathcal{A})$ in $B_{\mathcal{C}}(E)$:

$$\mathcal{B} = \{ s \in B_{\mathcal{C}}(E) : s\alpha(\mathcal{A}) \subseteq \alpha(\mathcal{A}),\ \alpha(\mathcal{A})s \subseteq \alpha(\mathcal{A}) \}.$$

Then α extends to a $*$-isom. between $M(\mathcal{A})$ and B.

<u>Proof</u>: By definition of $\mathcal{B}$, $\alpha(\mathcal{A})$ is an ideal of $\mathcal{B}$. In fact, $\alpha(\mathcal{A})$ is essential in $\mathcal{B}$. Indeed, if $b \in \mathcal{B}$ and $b\alpha(\mathcal{A}) = \{0\}$, then $b\alpha(\mathcal{A})E = \{0\}$. Hence, $bE = \{0\}$ by non-degeneracy of $\alpha(\mathcal{A})$ in $B_{\mathcal{C}}(E)$. Thus, $b = 0$. Now, we'll show $\mathcal{B}$ is a maximal essential extension. Suppose $\mathcal{A}$ is an essential ideal in a C^*-alg. D. Then by Prop. 2.1, α extends to $\bar{\alpha} : D \to B_{\mathcal{C}}(E)$, which is an injective $*$-h'ism. Moreover, $\bar{\alpha}(D) \subseteq \mathcal{B} \subseteq B_{\mathcal{C}}(E)$. Thus, D embeds into $\mathcal{B}$. Now, $\mathcal{B}$ has the maximal essential extension, so by Theorem 2.2, $\mathcal{B} \cong B(\mathcal{A})$ $*$-isom. $\boxtimes$

<u>Theorem 2.4</u> (Lance): If E is a Hilbert $\mathcal{A}$-module, then $B_{\mathcal{A}}(E) \cong M(K(E))$. <u>Kasparov</u>

(This is sort of just extending the result of $B(\mathcal{A}) \cong M(\mathcal{A}) \cong M(K(\mathcal{A}))$ to all Hilbert $\mathcal{A}$-modules).

<u>Proof</u>: $K(E) \xhookrightarrow{i} B_{\mathcal{A}}(E)$ is non-degenerate, and since $K(E)$ is already an ideal in $B(E)$, $B(E)$ <u>is</u> the idealiser of $i(K(E)) = K(E)$. By Proposition 2.3, $B_{\mathcal{A}}(E) \underset{*}{\cong} M(K(E))$. $\boxtimes$

Applications

1) If $\mathcal{A} = \mathbb{C}$, then for a Hilbert space H,

$$M(K(H)) = B(H).$$

2) Let X be a locally compact Hausdorff space. Then $M(C_0(X)) \cong C(\beta X)$ where βX is Stone-Čech compactification. Notice

$$\begin{aligned} \pi : C_0(X) &\longrightarrow B(L^2(X)) \\ f &\longmapsto M_f \end{aligned}$$

where π is just a restriction of $\hat{\pi} : L^\infty(X) \to B(L^2(X))$. We'll show the idealiser of $C_0(X)$ in $B(L^2(X))$ is $C_b(X)$.

($\subseteq$): Clear.

($\supseteq$): Let T be in idealiser of $C_0(X)$. For each $f \in C_0(X)$, $\exists g, h \in C_0(X)$ s.t.

$$T M_f = M_h \text{ and } M_f T = M_g.$$

Choose a "pre-compact" cover $X = \bigsqcup_{i \in I} U_i$. By Urysohn, $\exists f_i : X \to [0,1]$ s.t. $f_i|_{\overline{U_i}} = 1$ and $f|_{V_i^c} = 0$ where $V_i > U_i$ is an open set. Then $\exists g_i$ s.t. $M_{f_i} T = M_{g_i}$

$\rightsquigarrow$ $T = M_{\sum_{i \in I} g_i}$ (lots of little details).

Def'n: Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-alg. A morphism from $\mathcal{A}$ to $\mathcal{B}$ is a nondegenerate $*$-h'ism from $\mathcal{A}$ to $M(\mathcal{B})$. Denote all morphisms by $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$.

$Mor(A,B)$ is larger than $\{\varphi : A \to B \mid \varphi \text{ } *\text{-h'ism}\}$ but we "pay the price" of nondegeneracy. So, we want to know some equivalent conditions for when a map is nondegenerate.

__Proposition 2.5__ (Lance)

Let A, B be C^*-algebras, E a Hilbert B-module. If $\alpha : A \to B_B(E)$, TFAE:

(i) α is nondegenerate

(ii) $\alpha = \bar{\alpha}|_A$ where $\bar{\alpha} : M(A) \to B_B(E)$ is a unital $*$-h'ism and $\bar{\alpha}$ is strictly cts. on the unit ball.

(iii) for any approximate unit (e_i) of A, $\alpha(e_i) \to id_E$ strictly.

__Proof__: (i) $\to$ (ii) By Proposition 2.1, $\exists$ unital extension $\bar{\alpha} : M(A) \to B_B(E)$ of α. Suppose $x_i \to x$ strictly in the unit ball of $M(A)$. Then $\|(x_i - x)a\| \xrightarrow{i} 0$ $\forall a \in A$. Hence, $\forall a \in A$, $\xi \in H$,

$$\|\bar{\alpha}(x_i - x)\alpha(a)\xi\| = \|\alpha((x_i - x)a)\xi\| \xrightarrow{i} 0$$

because α is continuous (it's a $*$-h'ism of C^*-alg.).

(ii) $\to$ (iii) Let (e_i) be an approx. unit for A. Then by Prop. 1.3, $e_i x \to x$ strictly $\forall x \in A$. Hence, $e_i \to id_A$ strictly. As $\bar{\alpha}$ is a (continuous) unital $*$-h'ism, $\alpha(e_i) \to id_E$ strictly in $B_B(E)$.

(iii) $\to$ (i) Sps. $\alpha(e_i) \to id_E$ strictly. Let $h \in E$. Then $\alpha(e_i)h \to h$. Since $\{\alpha(e_i)h\}_{i \in I} \subseteq \alpha(A)E$, $E \subseteq \overline{\alpha(A)E}$, so α is nondegenerate. $\square$

For $\alpha \in Mor(A,B)$, just denote $\overline{\alpha}: M(A) \to M(B)$ by α (extension from Prop. 2.5). Let $\alpha \in Mor(A,B)$, $\beta \in Mor(B,C)$.

we want $\beta\alpha : M(A) \to M(C)$ to be in $Mor(A,C)$.

It's a $*$-h'ism. To show it's nondegenerate, we'll show $\beta\alpha$ satisfies Prop. 2.5 (ii). α is a restriction of $\overline{\alpha}: M(A) \to M(B)$ and β is a restriction of $\overline{\beta}: M(B) \to M(C)$ where $\overline{\alpha}$ and $\overline{\beta}$ are unital $*$-h'isms strictly cts. ~~on unit balls of~~ of $M(A)$ and $M(B)$, respectively. Their composition is strictly cts. on unit ball of $M(A)$, so $\beta\alpha \in Mor(A,C)$.

Let A, B be C^*-algebras, F a B-module. By Theorem 2.4, $B_B(F) \cong M(K_B(F))$. Also, $Mor(A, K(F))$ are nondegenerate $*$-h'isms from A to $B(F)$.

Chapter 3: Projections & Unitaries

Recall, if E is a Hilbert C^*-module, $F \subseteq E$ closed subspace, then $F^\perp$ is easily defined, but $F \oplus F^\perp \neq E$ in general. If F does satisfy this, we say F is (orthogonally) complemented. In contrast, if $G \subseteq E$ is a closed subspace then G is topologically complemented if $\exists$ a closed subspace $G' \subseteq E$ s.t. $G \cap G'$ and $G \oplus G'$. Note: orth. comp $\Rightarrow$ top. comp.

Example: $A = C[0,1]$, $J = \{f \in A : f(0) = 0\}$ is a closed ideal of A (and thus is a subspace). Define $E := A \oplus J$. Then, take $F := \{(f,f) : f \in J\}$ which is a closed A-submodule of E. Then,

$$F^\perp = \{(g,h) \in E : \langle (g,h), (f,f) \rangle = 0 \ \forall f \in J\}$$
$$= \{(g,h) \in E : \bar{g}f + \bar{h}f = 0 \ \forall f \in J\}$$
$$= \{(g,-g) : g \in J\}$$

Note $F \oplus F^\perp = \{(f+g, f-g) : f, g \in J\} = J \oplus J \subsetneq A \oplus J = E$.
However, $G = \{(f,0) : f \in A\}$ satisfies $F \cap G = \{(0,0)\}$ and $G + F = E$. So, F is topologically complemented but is not orthogonally complemented.

maximal ideals?

Projections

*If $F \subseteq E$ is orth. comp., we can write $E \ni x = \underset{\in F}{x_1} + \underset{\in F^\perp}{x_2}$.
Define $p_i : E \to E$ by $x \mapsto x_i$. Then p is adjointable, is self-adjoint, and idempotent.

* On the other hand, if $G \subseteq E$ is top. comp., define $g: E \to E$ by $x = \underset{\in G}{x_1} + \underset{\in G'}{x_2} \longmapsto x_1 \in G$. Then g is idempotent, linear, and bounded, but need not be adjointable!

<u>EX</u>: $A = C[0,1]$, $J = \{f \in A : f(0) = 0\}$, $E = A \oplus J$. Define $g(f,g) := (f-g, 0)$. Then g is linear, bdd., idempotent, and range$(g) = G$. But, it's not adjointable!

<u>Proof</u>: Suppose g^* exists. Then $\forall f_i \in A$, $g_i \in J$,

$$\langle g(f_1,g_1),(f_2,g_2)\rangle = \langle (f_1,g_1), \overbrace{g^*(f_2,g_2)}^{=(f_3,g_3)\text{ for}}\rangle \quad \text{some } f_3 \in A,\ g_3 \in J$$

$$\shortparallel$$

$$\langle (f_1 - g_1, 0), (f_2, g_2)\rangle$$

$$\shortparallel$$

$$\overline{f_1 - g_1} \cdot f_2 = \overline{f_1} f_2 - \overline{g_1} f_2 = \overline{f_1} f_3 + \overline{g_1} g_3$$

If $g_1 = 0$, then $\overline{f_1} f_2 = \overline{f_1} f_3$, so if $f_1 = 1$, $f_2 = f_3$.
Similarly, if $f_1 = 0$, $-\overline{g_1} f_2 = \overline{g_1} g_3$. So long as $g \neq 0$ for $x > 0$ anywhere, we conclude $g_3 = -f_2$, on $(0,1]$.
But then by continuity, $g_3(0) = -f_2(0)$. But $-f_2 \notin J$ necessarily

<u>Lemma</u>: Suppose E is a Hilbert A-module, $t \in B_A(E)$ s.t. $t = t^*$ and $\|tx\| \geq k\|x\|$ $\forall x \in E$ for some $k > 0$. Then t is invertible in $B(E)$.

* In $B(H)$, both t bdd. below AND ran t dense is required for invertibility

Proof: It's enough to show $0 \notin \sigma(t)$. BWOC, sps. $0 \in \sigma(t)$.
Let $f \in C(\mathbb{R})$ s.t. $f(\underset{\in \sigma(t)}{0}) = 1 = \|f\|_\infty$ and $f(\lambda) = 0 \ \forall |\lambda| \geq \frac{1}{2}k$
by Urysohn's Lemma. Let $s = f(t)$ (functional calculus)
then $\|s\| = 1$, because Gelfand map is isometric. Similarly,
$\|ts\| \leq \frac{1}{2}k$.

* $s = 0$, done.
* $s \neq 0$, $\|tf(t)\| = \|xf(x)\|_\infty \leq$ ✓.

Choose $x \in E$ s.t. $\|x\| = 1$ and $\|sx\| \geq 1/2$. Then,

$$\|tsx\| \leq \frac{1}{2}k < k\|sx\|.$$

which contradicts bounded below w/ t applied to sx. ☐

Theorem (due to Miščenko): E, F are Hilbert $\mathcal{A}$-modules, $t \in B(E,F)$. Suppose t has closed range. Then,

(i) $\ker(t)$ is complemented by $\operatorname{ran}(t^*)$ (in E)

(ii) $\operatorname{ran}(t)$ is complemented by $\ker(t^*)$ (in F)

(iii) $t^* \in B(F,E)$ has closed range.

Proof: Assume t is surjective ($t \in B(E, \operatorname{ran} t)$ is onto).
By the open mapping theorem, $t(E_1)$ is open. Also, (E_1: unit ball of E)
$0 \in t(E_1)$, so $t(E_1)$ contains an open nbhd U of 0 s.t.
$\overline{U} = B_\delta(0) \subset t(E_1)$.

Let $y \in F$, let $z \in E$ s.t. $tz = y$ (surj.). Set $\underline{x := \frac{\|y\|}{\delta} z}$. ?

Then, $\exists x$ s.t. $tx = y$, $\|x\| \le \|y\|/\delta$. Hence,

$$\|t^*y\|_E^2 = \|\langle t^*y, t^*y\rangle\|_A^2 \Rightarrow \|y\|_E^2 = \|\langle y, y\rangle\|_A^2$$

$$= \|\langle y, tt^*y\rangle\|^2 \qquad = \|\langle tx, y\rangle\|_A^2$$

$$\le \|y\| \cdot \|tt^*y\| \qquad = \|\langle x, t^*y\rangle_A\|^2$$

$$\le \|x\| \cdot \|t^*y\| \le \frac{\|y\|}{\delta}\|t^*y\| \le \frac{\|y\|}{\delta} \cdot \|y\|^{1/2} \cdot \|tt^*y\|^{1/2}$$

$$\rightsquigarrow \|y\|\delta \le \|tt^*y\|.$$

Hence, by the Lemma, $tt^* \in B(E)$ is invertible. Let $z \in E$. Then $tz = tt^*w$ for some $w \in F$, hence

$$t(z - t^*w) = 0 \Rightarrow z - t^*w \in \ker t$$

Now, $z = (z - t^*w) + t^*w \in \ker t + \operatorname{ran} t^*$, so $\operatorname{ran} t^* = (\ker t)^\perp$.

For (ii), (iii), let $F_0 = t(E) \subset F$. Let $\hat{t}$ be t in $B(E, F_0)$.

We'll show $\operatorname{ran}(\hat{t}^*) = \operatorname{ran}(t^*)$. By (i),

$(\ker t)^\perp = \operatorname{ran}(\hat{t}^*)$.

Also, $t^*|_{F_0} = \hat{t}^*$. Indeed, for $y \in F_0$, $\forall x \in E$

$$\langle x, t^*y\rangle = \langle tx, y\rangle = \langle \hat{t}x, y\rangle = \langle x, \hat{t}^*y\rangle.$$

Clearly, $\operatorname{ran}(\hat{t}^*) \subseteq \operatorname{ran} t^*$, but $\operatorname{ran}(t^*) \subseteq \ker(t)^\perp \underset{(i)}{=} \operatorname{ran}(\hat{t}^*)$.

As $\hat{t}^*$ has closed range, (it equals $(\ker t)^\perp$) so does t^*. ∎

Therefore, $\ker t^*$ and $\operatorname{ran} t^*$ are also complemented whenever t has closed range.

<u>Example</u> (no complements).

Let A be $C[-1,1]$ and view as a Hilbert C^*-module over itself. Define $t: A \to A$ by (<u>Note</u>: $t = t^*$)

$$(tf)(x) = \begin{cases} x f(x), & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Then $\ker t = \{ f \in A : f(x) = 0 \ \forall x \geq 0 \}$, $\overline{\operatorname{ran} t} = \{ f \in A : f(x) = 0 \ \forall x < 0 \}$. needs to be closed!

However, $(\ker t)^\perp = \overline{\operatorname{ran} t}$ and $(\overline{\operatorname{ran} t})^\perp = \ker t$, but $\ker t + \overline{\operatorname{ran} t} \neq E$ because $f(x) = 1 \ \forall x \in [-1,1]$ is not there.

<u>UNITARIES</u>

These are better!

<u>Theorem</u>: Let A be a C^*-alg., E and F Hilbert A-modules. Then, TFAE:

(i) u is isometric, surjective, A-linear

(ii) u is a unitary in $B(E,F)$, i.e., $u^*u = id_E$, $uu^* = id_F$.

<u>Proposition</u>: If E, F are Hilbert A-modules, let $w: E \to F$ be A-linear. Then, TFAE:

(i) w is isometric ($\forall x, y \in E$, $\langle wx, wy \rangle = \langle x, y \rangle$) and $\operatorname{ran}(w)$ is complemented.

(ii) $w \in B(E,F)$ and $w^* w = id_E$.

<u>Proposition</u>: Let $t \in B(E,F)$. Then $\overline{t^* F} = \overline{t^* t E}$.

(In <u>regular Hilbert space</u>, <u>both are equal</u> to $(\ker t)^\perp$).

"<u>Proof</u>:" Let $J = \{ t^* t u : u \in B(E) \}$, $K = \{ t^* v : v \in B(E,F) \}$, both of <u>which</u> are closed ideals in $B(E)$. We show $\overline{t^* t E} = \overline{JE}$ and $\overline{t^* F} = \overline{KE}$. Show $J = K$. By defn, $J \subseteq K$.

For $K \subseteq J$, Show $\forall$ states ρ on $B(E)$ s.t. $\rho(J)=0$, we have $\rho(K)=0$. By a lemma, if these two closed ideals were proper, there would be a state separating them. $\boxtimes$

Interlude: Tensor Products

Algebraic: X, Y vector spaces, $X \times Y$ discrete topology.
Consider $C_c(X \times Y)$ = finitely supported b/c discrete.
Note: $\{\chi_{(x,y)} : x \in X, y \in Y\}$ is a basis for $X \times Y$.

Let K be the span of elts. of the form

① $\chi_{(x_1+x_2,y)} - \chi_{(x_1,y)} - \chi_{(x_2,y)}$

② $\chi_{(x,y_1+y_2)} - \chi_{(x,y_1)} - \chi_{(x,y_2)}$

③ $\chi_{(\lambda x,y)} - \lambda\chi_{(x,y)}$

④ $\chi_{(x,\lambda y)} - \lambda\chi_{(x,y)}$

The algebraic tensor product $X \odot Y = C_c(X \times Y)/K$.
Notation: $x \otimes y = \chi_{(x,y)} + K$ (The set $\{\chi_{(x,y)} : x \in X, y \in Y\}$ loses linear independence in the quotient, so the cosets $\{\chi_{(x,y)} + K : x \in X, y \in Y\}$ are a spanning set for $X \odot Y$, but it's not a basis. Also, K gives

- $(x_1+x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$
- $x \otimes (y_1+y_2) = x \otimes y_1 + x \otimes y_2$
- $(\lambda x) \otimes y = \lambda(x \otimes y) = x \otimes (\lambda y)$.

Universal Property

If Z is a v.s., $\sigma : X \times Y \longrightarrow Z$ is bilinear, then $\exists !$ linear map $\bar{\sigma} : X \odot Y \longrightarrow Z$ s.t. the following diagram commutes:

$$\begin{tikzcd}
X \times Y \arrow[r, "\sigma"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \\
X \odot Y \arrow[ur, dashed, "\bar{\sigma}"'] &
\end{tikzcd}$$

Corollary: *If $\varphi : W \longrightarrow Y$, $\psi : X \longrightarrow Z$ are linear, $\exists !$ linear map $\varphi \odot \psi : W \odot X \longrightarrow Y \odot Z$ such that $(\varphi \cdot \psi)(w \otimes x) = \varphi(w) \otimes \psi(x)$.

* If $Y = Z = C$, an algebra, then $\exists !$ linear map

$$\varphi \times \psi : W \odot X \longrightarrow C \ \text{ s.t. } (\varphi \times \psi)(w \otimes x) = \varphi(w)\psi(x).$$

* If W & X are $*$-algebras & φ and ψ are $*$-h'isms, then $\varphi \odot \psi$ is a $*$-h'ism.

* If φ, ψ are linear fnc'ls, so is $\varphi \times \psi$.

Extra structure

If A, B are C^*-algebras, then $A \odot B$ has involution $(a \otimes b)^* = a^* \otimes b^*$ (extend linearly) and multiplication

$$\left(\sum_{i=1}^{m} a_i \otimes b_i\right)\left(\sum_{j=1}^{n} c_j \otimes d_j\right) = \sum_{i,j} a_i c_j \otimes b_i d_j .$$

Note: • If $\{x_i\}$ is a basis for X, $\{y_i\}$ basis for Y, then $\{x_i \otimes y_i\}$ is a basis for $X \odot Y$.

• $\{x_\alpha\}$ basis for X, $\Rightarrow$ for each $v \in X \odot Y$, $\exists !$ $\{y_\beta\} \subseteq Y$ s.t. $v = \sum x_i \otimes y_i$

at most finitely many

Hilbert Spaces

H, K Hilbert spaces, define

$$\left\langle \sum a_i \otimes b_i, \sum x_j \otimes y_j \right\rangle_{H \otimes K} := \sum_{i,j} \langle a_i, x_j \rangle_H \langle b_i, y_j \rangle_K$$

In particular, $\|h \otimes k\| = \|h\| \cdot \|k\|$. Define $H \bar{\otimes} K$ to be the completion of $H \odot K$ with respect to $\|\cdot\|_{H \otimes K}$.

Proposition: $S \in B(H)$, $T \in B(K)$, then $\exists!$ $S \otimes T \in B(H \otimes K)$ s.t. $(S \otimes T)(h \otimes k) := Sh \otimes Tk$. Moreover, $\|S \otimes T\| = \|S\| \cdot \|T\|$.

Quantum Groups

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-alg's. $T: \mathcal{A} \to \mathcal{B}$ is an anti-homomorphism if it's linear and $T(ab) = T(b)T(a)$, $a, b \in \mathcal{A}$. Let $i: \mathcal{A} \to \mathcal{A}$ be the identity, $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ be mult: $m(a \otimes b) = ab$.

<u>Defn</u>: Let $\mathcal{A}$ be a fin.-dim'l C^*-alg. $\mathcal{A}$

- <u>coproduct</u> on $\mathcal{A}$ is a unital $*$-h'ism

$$\Phi: \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} \quad \text{s.t.}$$

$$(\Phi \otimes i) \circ \Phi = (i \otimes \Phi) \circ \Phi \qquad \text{(coassociativity)}$$

from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

"multiplication"

- <u>counit</u> is a unital $*$-h'ism $\varepsilon: \mathcal{A} \longrightarrow \mathbb{C}$ s.t. $(\varepsilon \otimes i) \circ \Phi = i = (i \otimes \varepsilon) \circ \Phi$. from $\mathcal{A}$ to $\mathcal{A}$.

"identy"

- <u>antipode</u> is a unital antihomomorphism $K: \mathcal{A} \longrightarrow \mathcal{A}$ s.t. $m \circ (K \otimes i) \circ \Phi = \varepsilon \cdot I_{\mathcal{A}}$ from $\mathcal{A}$ to $\mathcal{A}$ $= m \circ (i \otimes K) \circ \Phi$.

"inverse operation"

A <u>finite quantum gp.</u> is a fin. dim'l C^*-alg. with a coproduct, counit, and antipode.

Remarks: If we assume K is linear, its anti'h'ism property follows from associativity condition. Also, $K(K(a)^*)^* = a \ \forall a \in \mathcal{A}$, so K is ~~[illegible]~~ surjective.

Example: Let G be a finite group w/ unit $e \in G$. Then $\mathcal{A} = C(G)$ is a finite quantum gp w/:

$$\Phi : C(G) \to C(G) \otimes C(G) \quad (\cong C(G \times G))$$

$$(\Phi f)(x, y) = f(xy)$$

$$\varepsilon : C(G) \longrightarrow \mathbb{C}, \qquad \varepsilon(f) := f(e)$$

$$K : C(G) \longrightarrow C(G) \qquad K(f)(x) = f(x^{-1})$$

Check when $\Phi f = g_1 \otimes g_2$

$$\left[(\Phi \otimes i) \circ \Phi\right](f)^{(x,y,z)} = (\Phi \otimes i)(g_1 \otimes g_2)(x, y, z)$$

$$= (\Phi g_1 \otimes g_2)(x, y, z)$$

$$= (\Phi g_1)(x, y) \otimes g_2(z)$$

$$= g_1(xy) g_2(z) \overset{??}{=} g_1(x) g_2(yz).$$

$= ~~[illegible]~~ f(xyz).$

$\ast$ Φ must be given by comp. w/ $\phi : G \times G \longrightarrow G$.

<u>Theorem</u>: Let $(A, \Phi, \varepsilon, \kappa)$ be a finite quantum group. If A is abelian, then $(A, \Phi, \varepsilon, \kappa)$ is isomorphic to the quantum gp. $C(G)$ for some finite gp. G.

<u>Proof</u>: Φ gives associative binary op., ε gives identity, κ gives inverse. (See Weaver).

<u>Ex</u>: Let G be a finite gp, and for $x \in G$, define unitary translation op. $T_x : \ell^2(G) \to \ell^2(G)$ by $(T_x f)(y) = f(x^{-1}y)$. Then

$$\begin{aligned}(T_x(T_y)(f)(z)) &= (T_y f)(x^{-1}z) \\ &= f(y^{-1}x^{-1}z) \\ &= T_{xy} f(z)\end{aligned}$$

Let $C^*(G)$ ~~by~~ be C^*-gen. by $\{T_x : x \in G\}$. (left regular representation).

Define

$$\Phi(\chi_x) := \chi_x \otimes \chi_x$$

$$\varepsilon(\chi_x) := \langle \chi_x, \chi_e \rangle$$

$$\kappa(\chi_x) := \chi_{x^{-1}}$$

Then $C^*(G)$ is a finite quantum gp.