# It's a Noncommutative World After All

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# Outline

1. Noncommutative Happenings in Everyday Life
2. A Quantum System
    - The Double Slit Experiment
    - Quantum Measurements via Integration
3. Probability turned Quantum
4. The Heisenberg Uncertainty Principle

# Noncommutative Happenings in Everyday Life

**Commutative Operations:** For any real numbers $a$ and $b$,

$$a + b = b + a \text{ and } ab = ba.$$

"Addition and multiplication are commutative."

**Noncommutative Operations:**

- Getting dressed
- Doing laundry
- Function composition

**Example**

If $f(x) = x^2$ and $g(x) = x + 1$, does $f(g(x)) = g(f(x))$?

$$f(g(x)) = (x + 1)^2 \text{ and } g(f(x)) = x^2 + 1.$$

# Examples of Noncommutative Operations

- Getting dressed
- Doing laundry
- Function composition

**Example**

If $f(x) = x^2$ and $g(x) = x + 1$, does $f(g(x)) = g(f(x))$?

**No:** $(x+1)^2 \neq x^2 + 1$.

$\Rightarrow$ $f$ and $g$ do not commute under *function composition*:

$$f \circ g \neq g \circ f.$$

# Composition of Function Transformations

Function *transformations* are functions on functions.

- **Translations**: Apply $T_c$ to a function $f$ by $[T_c f](x) = f(x) + c$.
- **Scaling**: Apply $S_d$ to a function $f$ by $[S_d f](x) = df(x)$.

**Example**

Do $T_c$ and $S_d$ commute with respect to function composition? That is, for any function $f$, does

$$(T_c \circ S_d)f = (S_d \circ T_c)f?$$

Consider $c = 1$ and $d = 2$.

`https://www.desmos.com/calculator/axkqbn7se3`

# Composition of Function Transformations

- **Multiplication**: Apply $Q$ to $f$ by $[Qf](x) = xf(x)$.
- **Differentiation**: Apply $P$ to $f$ by $[Pf](x) = f'(x) \cdot i = if'(x)$.

**Example**

Do $Q$ and $P$ commute with respect to function composition? That is, for any function $f$, does

$$(Q \circ P)f = (P \circ Q)f?$$

$$(P \circ Q)f - (Q \circ P)f = if$$

The transformations $P$ and $Q$ are known as the "momentum" and "position" transformations. This relation is known as the

Heisenberg Commutation Relation.

# The Double Slit Experiment

1. Shoot electrons out of a (stationary) electron beam at a screen with two slits cut in it.
2. A second screen (behind the first) detects where the electrons land.

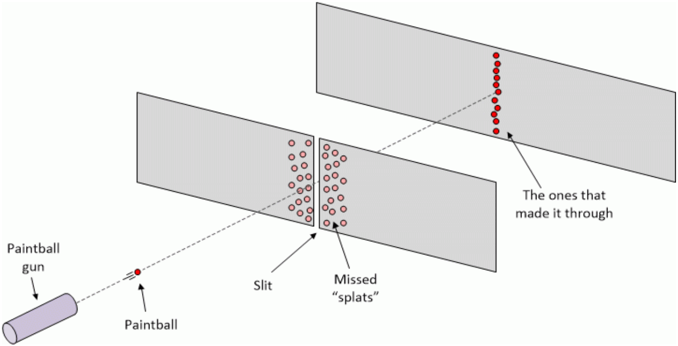


Figure: If electrons were paintballs.

# The Double Slit Experiment

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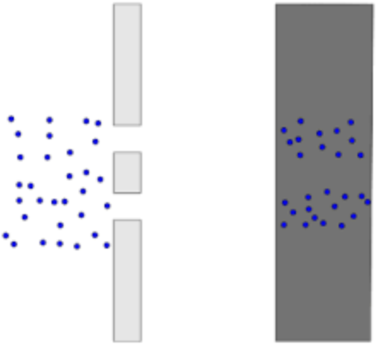

Figure: Paintballs through two slits.

# The Double Slit Experiment

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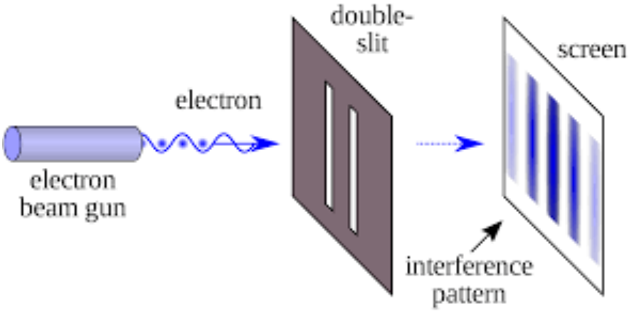



Figure: Actual Double Slit Experiment.

# The Double Slit Experiment

**Conclusions:** Electrons are not paintballs, nor do they behave like them. Instead, electrons (and other quantum particles) behave much like waves—unpredictably. But, they do behave probabilistically.

A particle's motion is represented by a function, $\psi$, that contains *probabilistic* information.

1. We call $\psi$ the state of the particle.
2. We require $\psi(x) \cdot \psi(x) = \psi(x)^2$ to be a *probability density function*:

$$\int_{-\infty}^{\infty} \psi(x)^2 \, dx = 1.$$

How do we compare two states $\psi$ and $\phi$?

$$\langle \psi, \phi \rangle = \int_{-\infty}^{\infty} \psi(x) \cdot \phi(x) \, dx.$$

Think of the *dot product* of two unit vectors:

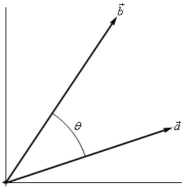



Figure: $\mathbf{a} \cdot \mathbf{b} = \cos \theta$.

$\Rightarrow \langle \psi, \phi \rangle$ is measuring some geometric difference between $\psi$ and $\phi$.

https://www.desmos.com/calculator/utix5squrb

# Quantum Measurements

Why did we call $Q$ the position transformation?

- If $\psi(x)^2$ is the p.d.f. for position, the *expected value* is

$$\int_{-\infty}^{\infty} x\psi(x)^2 \, dx = \int_{-\infty}^{\infty} x\psi(x) \cdot \psi(x) \, dx = \langle Q\psi, \psi \rangle .$$

- Let $E(Q) := \langle Q\psi, \psi \rangle$ .

Let's use this same idea for momentum:

$$E(P) = \langle P\psi, \psi \rangle = \int_{-\infty}^{\infty} i\psi'(x) \cdot \psi(x) \, dx.$$

What's the idea behind the *expected value*?

# Expected Values

## Example

Consider a *probability mass function* $\psi(x)^2 = p(x)$ on the positions $x = \{0, 1, 2\}$ given by

$$p(0) = 0.5 \qquad p(1) = 0.3 \qquad p(2) = 0.2.$$

Note: $\sum_{n=0}^{2} p(n) = 1$.

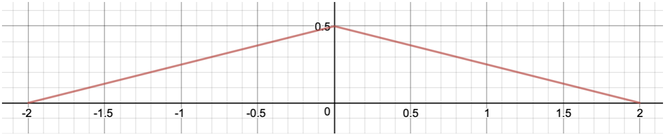
Then the expected value of the position is

$$E(Q) = \sum_{n=0}^{2} np(n) = 0p(0) + 1p(1) + 2p(2) = 0(0.5) + 1(0.3) + 2(0.2) = 0.7.$$

So an expected value is a *weighted average*.

## Example

Let $p(x) = \frac{1}{2}(1 - \left|\frac{1}{2}x\right|)$ for $-2 \leq x \leq 2$, and $p(x) = 0$ otherwise.



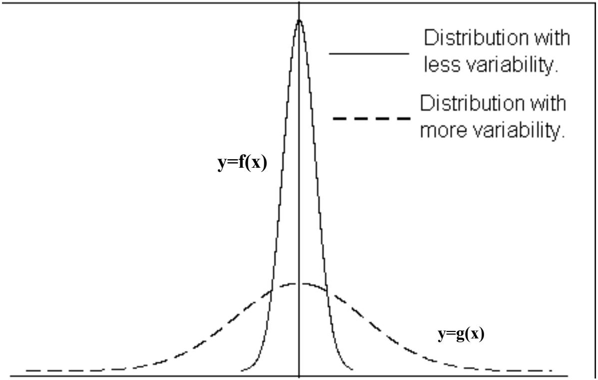

- Note: $\int_{-\infty}^{\infty} p(x)\, dx = 1$.

Where do you expect to find the particle?

$$E(Q) = \int_{-\infty}^{\infty} xp(x)\, dx = 0.$$

How certain are you that you'll find it near $x = 0$?

# More Ideas from Probability




**Moral:** Less spread in probability relates to higher certainty.
**Goal:** Minimize the *variance*, the square of the *standard deviation*

$$\sigma^2 = E(Q^2) - E(Q)^2.$$

# Simultaneous Pursuit for Certainty

Can we simultaneously improve our certainty of $E(Q)$ *and* $E(P)$?

- Let $\tau^2 = E(P^2) - E(P)^2$ be the variance for $P$.
- Let's try to simultaneously *minimize* $\sigma^2$ and $\tau^2$ by minimizing $\sigma^2\tau^2$.

Let's be clever!

$$\sigma^2 = E(Q^2) - E(Q)^2 = E(Q^2) - 2E(Q)^2 + E(Q)^2$$

$$= \int_{-\infty}^{\infty} x^2\psi(x)^2\,dx - 2E(Q)\int_{-\infty}^{\infty} x\psi(x)^2\,dx + E(Q)^2\int_{-\infty}^{\infty} \psi(x)^2\,dx$$

$$= \int_{-\infty}^{\infty} x^2\psi(x)^2 - 2E(Q)x\psi(x)^2 + E(Q)^2\psi(x)^2\,dx$$

$$= \int_{-\infty}^{\infty} \left[x^2 - 2E(Q)x + E(Q)^2\right]\psi(x)^2\,dx = \int_{-\infty}^{\infty} [x - E(Q)]^2\,\psi(x)^2\,dx$$

# Simultaneous Pursuit for Certainty

So

$$\sigma^2 = \int_{-\infty}^{\infty} [x - E(Q)]^2 \, \psi(x)^2 \, dx = \int_{-\infty}^{\infty} \hat{Q}^2 \psi(x)^2 \, dx.$$

Similarly,

$$\tau^2 = \int_{-\infty}^{\infty} \hat{P}^2 \psi(x)^2 \, dx.$$

Now,

$$\sigma^2 = \int_{-\infty}^{\infty} \hat{Q}\psi(x) \cdot \hat{Q}\psi(x) \, dx = \left\langle \hat{Q}\psi, \hat{Q}\psi \right\rangle$$

and

$$\tau^2 = \int_{-\infty}^{\infty} \hat{P}\psi(x) \cdot \hat{P}\psi(x) \, dx = \left\langle \hat{P}\psi, \hat{P}\psi \right\rangle.$$

# Simultaneous Pursuit for Certainty

Thus far, we've reduced the product of the variances $\sigma^2$ and $\tau^2$ to

$$\sigma^2\tau^2 = \left\langle \hat{Q}\psi, \hat{Q}\psi \right\rangle \left\langle \hat{P}\psi, \hat{P}\psi \right\rangle .$$

Recall the *dot product* of a vector $\mathbf{a}$ with itself is $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$. Similarly,

$$\sigma^2\tau^2 = \left\| \hat{Q}\psi \right\|^2 \left\| \hat{P}\psi \right\|^2 .$$

Again, using the dot product for guidance,

$$\|\mathbf{a}\| \, \|\mathbf{b}\| \geq |\|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta| = |\mathbf{a} \cdot \mathbf{b}|$$

$$\Rightarrow \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \geq |\mathbf{a} \cdot \mathbf{b}|^2$$

$$\Rightarrow \sigma^2\tau^2 = \left\| \hat{Q}\psi \right\|^2 \left\| \hat{P}\psi \right\|^2 \geq \left| \left\langle \hat{Q}\psi, \hat{P}\psi \right\rangle \right|^2 .$$

# Simultaneous Pursuit for Certainty

The inequality $\sigma^2\tau^2 \geq \left|\left\langle \hat{Q}\psi, \hat{P}\psi \right\rangle\right|^2$ *still* depends on $\psi$. Note that

$$\left\langle \hat{Q}\psi, \hat{P}\psi \right\rangle = \left\langle (\hat{P} \circ \hat{Q})\psi, \psi \right\rangle .$$

Now, $\sigma^2\tau^2 \geq \left|\left\langle (\hat{P} \circ \hat{Q})\psi, \psi \right\rangle\right|^2$. Let's be clever again!

$$\hat{P} \circ \hat{Q} = \frac{1}{2}(\hat{P} \circ \hat{Q}) + \frac{1}{2}(\hat{P} \circ \hat{Q}) = \frac{1}{2}(\hat{P} \circ \hat{Q} + \hat{Q} \circ \hat{P}) + \frac{1}{2} i.$$

$$\Rightarrow \sigma^2\tau^2 \geq \left|\left\langle (\hat{P} \circ \hat{Q})\psi, \psi \right\rangle\right|^2 = \left|\left\langle \left[\frac{1}{2}(\hat{P} \circ \hat{Q} + \hat{Q} \circ \hat{P}) + \frac{1}{2} i\right] \psi, \psi \right\rangle\right|^2$$

$$= \left|\left\langle \frac{1}{2}(\hat{P} \circ \hat{Q} + \hat{Q} \circ \hat{P})\psi, \psi \right\rangle + i \left(\frac{1}{2} \langle \psi, \psi \rangle\right)\right|^2$$

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$$= \left|\left\langle \frac{1}{2}(\hat{P} \circ \hat{Q} + \hat{Q} \circ \hat{P})\psi, \psi \right\rangle + i\left(\frac{1}{2}\right)\right|^2$$

# Heisenberg Uncertainty Principle

Therefore, $\sigma^2\tau^2 \geq \frac{1}{2}^2 = \frac{1}{4}$. Although $\sigma^2$ and $\tau^2$ depend on $\psi$, the product of their variances is bounded below by $\frac{1}{4}$, *independent of* $\psi$.

This is known as the Heisenberg Uncertainty Principle.

`https://www.desmos.com/calculator/txeovzdmbl`

Thank you!