A NONCOMMUTATIVE WORLD

Introductory Examples

Set	Operation	Commutative	Noncommutative
$\mathbb{R}$	$+$	X	
$\mathbb{R}$	$\times$	X	
$\mathbb{R}^n$	$+$	X	
$\mathbb{R}^n$	$\times$	X	
Vectors in $\mathbb{R}^3$	cross product		$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
Functions on $\mathbb{R}$	$+$	X	
Functions on $\mathbb{R}$	$\times$	X	
Functions on $\mathbb{R}$	$\circ$		specific to the functions
Transformations of $\mathbb{R}^2$	$\circ$		X

Transformations of the xy-plane

Let's look at functions that map points on the plane to new points on the plane.

1. Translations: X_t moves a point (x, y) by t in the positive x-direction:

$$X_t(x, y) = (x + t, y).$$

   Similarly, Y_t moves a point (x, y) by t in the positive y-direction:

$$Y_t(x, y) = (x, y + t).$$

   Remark:

$$X_2(Y_{-3}(x, y)) = X_2(x, y-3) = (x+2, y-3) = Y_{-3}(x+2, y) = Y_{-3}(X_2(x, y)).$$

   In fact, for any horizontal shift s and vertical shift t,

$$X_s(Y_t(x, y)) = X_s(x, y+t) = (x+s, y+t) = X_s(x, y+t) = Y_t(X_s(x, y)),$$

   so X_s and Y_t commute under composition for any choice of s and t!

2. Rotations: Denote rotation of the plane by an angle of θ in the counterclockwise direction by U_θ. Some examples include:

$$U_{\pi/2}(x, y) = (-y, x), \quad U_\pi(x, y) = (-x, -y).$$

Remark: Rotations commute under composition! This is easier to see using 2×2 matrices, but we won't go into that here.

Question: Do translations and rotations commute under composition? No.

$$X_1(U_\pi(x,y)) = X_1(-x,-y) = (-x+1,-y),$$

while

$$U_\pi(X_1(x,y)) = U_\pi(x+1,y) = (-(x+1),y) = (-x-1,y).$$

3. Scales: We could scale the plane by a factor of c :

$$S_c(x,y) = (cx, cy).$$

4. Reflections: Denote reflection across the y axis by R_y and reflection across the x-axis by R_x, so $R_x(x,y) = (x,-y)$ and $R_y(x,y) = (-x,y)$.

Remark: While R_x and R_y do commute under composition, reflections and rotations do **not.**

$$R_x(U_{\pi/2}(x,y)) = R_x(-y,x) = (-y,-x),$$

while

$$U_{\pi/2}(R_x(x,y)) = U_{\pi/2}(x,-y) = (y,x).$$

Transformations of functions on $\mathbb{R}$

By moving points around in $\mathbb{R}^2$, we are intrinsically transforming functions. A function can be thought of in $\mathbb{R}^2$ by the set of points $(x, f(x))$.

1. Translations: If X_t moves a point (x,y) by t in the positive x-direction:

$$X_t(x, f(x)) = (x+t, f(x)).$$

The graph of the points $(x+t, f(x))$ is the same as the points $(x, f(x-t))$. So, X_t sends a function $f(x)$ to the new function $f(x-t)$.

Similarly, Y_t moves a point (x,y) by t in the positive y-direction: $Y_t(x, f(x)) = (x, f(x)+t)$. So, Y_t sends a function $f(x)$ to the new function $f(x)+t$.

2. Rotations: This doesn't really work for functions on the real line because many functions when rotated will no longer pass the vertical line test.

3. Scales: Since $S_c(x, f(x)) = (cx, cf(x))$, This is the same as all points $\left(x, cf\left(\frac{1}{c}x\right)\right)$.

4. Reflections: Recall $R_x(x, f(x)) = (x, -f(x))$, so R_x sends a function $f(x)$ to the new function $-f(x)$. Similarly, $R_y(x, f(x)) = (-x, f(x))$. This is the same as the set of all points $(x, f(-x))$, so R_y sends a function $f(x)$ to $f(-x)$.

5. Multiplication by another function Notice reflection across the x-axis is really just multiplication by the function $h(x) = -1$. I.e.,

$$R_x(f(x)) = -f(x) = h(x)f(x).$$

   Couldn't we let $g(x) = x^2$, and then take a function $f(x)$ and send it to the new function $g(x) \cdot f(x)$?

6. Derivatives Unlike the previous more geometric transformations of functions, we can think of sending a (differentiable) function to its derivative as a transformation. Let's call D this transformation:

$$D(f(x)) = f'(x).$$

Why all the fuss about changing our perspective from transformations on $\mathbb{R}^2$ *to transformations of functions on* $\mathbb{R}$? Well, as you can see in #5 and #6, there are a lot *more* flexibility in transformations we can define on functions than we can on the xy-plane. Also, quantum mechanics!

A basic example in QM

Experiments, such as the "Double Slit" experiment, tipped physicists off that particles do not move like little billiard balls, but like waves. In other words, it's really hard for us to know where they are and where they're headed.

Heisenberg Uncertainty Principle. We can never know with full certainty where a particle is and its velocity at the same time.

Suppose a particle, like an electron, is confined to move on the real line. I don't know where my particle will be at a fixed moment in time, I only know how likely it is for the particle to be in any given position on the real line. Perhaps my particle has a 10% chance of being at $x = 0$. Then, it can only have a 90% chance of being anywhere else on the real line at that same moment. The assignment of the other probabilities yields a function, $p(x)$, which tells us the probability that the particle is at x. For example, $p(0) = .1$. We need

$$\int_{-\infty}^{\infty} p(x)\, dx = 1.$$

This kind of a function is called a <u>probability density function</u>.

Example: NOT SMOOTH DRAWING OF PDF.
Example: SMOOTH DRAWING OF PDF.

Definition. 1 *The <u>expected value</u> of a probability density function $f(x)$ is given by the integral*

$$\int_{-\infty}^{\infty} x f(x)\, dx.$$

Example. 1 (Discrete!) *Suppose my particle has a 30% chance of being at 1, a 20% chance of being at 2, and a 50% chance of being at 3. I have a <u>probability mass function</u> $f(1) = .3$, $f(2) = .2$, and $f(3) = .5$. Then, my particle's expected position is*

$$1f(1) + 2f(2) + 3f(3) = 1(0.3) + 2(0.2) + 3(0.5) = 0.3 + 0.4 + 1.5 = 2.2.$$

So, the expected value of a function is a weighted average.

Example. 2 (Continuous!) *Suppose my particle has a p.d.f. given by*

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

This is called the standard normal distribution.
GRAPH!

Let's calculate the expected value of $f(x)$.

$$\begin{aligned}\int_{-\infty}^{\infty} xp(x)\,dx &= \int_{-\infty}^{\infty} x \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{b\to\infty} \int_{-b}^{b} xe^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{b\to\infty} \int_{b^2/2}^{b^2/2} e^{-u}\,du \\ &= \frac{1}{\sqrt{2\pi}} \lim_{b\to\infty} 0 \\ &= 0\end{aligned}$$

This makes sense! Look at my graph!

Remark: $p_a(x) := \frac{1}{\sqrt{2\pi}} e^{-(x-a)^2} 2$ is also a p.d.f. and has expected value a.

Like the position, the particle's velocity at a fixed moment in time is given by some other p.d.f., let's call it $v(x)$, so $\int_{-\infty}^{\infty} v(x)\,dx = 1$. The particle's expected velocity is then

$$\int_{-\infty}^{\infty} xv(x)\,dx.$$

Theorem. 1 (Fourier) *If $p(x)$ is the position p.d.f. for a particle at a fixed time and $v(x)$ is the velocity p.d.f. at that same fixed time, then*

$$\int_{-\infty}^{\infty} xv(x)\,dx = \int_{-\infty}^{\infty} p'(x)\,dx.$$

Heisenberg Uncertainty Principle

Note that $xf(x)$ is given by a transformation–let $h(x) = x$, so $f(x) \mapsto h(x)f(x)$ is this transformation. Let's call this transformation Q.
Given that $\int_{-\infty}^{\infty} xp(x)\,dx = \int_{-\infty}^{\infty} Q(p(x))\,dx$ gives the expected position and $\int_{-\infty}^{\infty} p'(x)\,dx = \int_{-\infty}^{\infty} D(p(x))\,dx$ gives the expected velocity of the particle, we can see how the function transformations Q and D can be seen as "measuring" the position and velocity of the particle.

If we want to know both the position and velocity of the particle, and we want to know both with certainty, we should be able to measure the position and then immediately following that, measure the velocity, and get the same result as measuring velocity and then immediately after that the position. What this would look like is:

$$(D \circ Q)f(x) = D(Q(f(x)) = D(xf(x)) = xf'(x) + f(x)$$

and

$$(Q \circ D)f(x) = Q(D(f(x)) = Q(f'(x)) = xf'(x).$$

How "far off" are these two measurements? Well,

$$(D \circ Q)f(x) - (Q \circ D)f(x) = xf'(x) + f(x) - xf'(x) = f(x).$$

So, unless $f(x) = 0$ for all x, which is no longer a probability density function, these two measurements are yielding a way different outcome!

My Research

In addition to the position and velocity of a particle, we want to know how much energy is stored up in this line from the particle's motion. To measure this, we use a transformation called the Hamiltonian, let's call it H.

The expected position and velocity can only be computed for single moments in time using the transformations Q and D. My research focuses on how quickly these expected values change with respect to time. What this requires is an understanding of this power series of operators:

$$Q + [H, Q]t + \frac{1}{2}[H, [H, Q]]t^2 + \frac{1}{3!}[H, [H, [H, Q]]]t^3 + ...$$

and

$$D + [H, D]t + \frac{1}{2}[H, [H, D]]t^2 + \frac{1}{3!}[H, [H, [H, D]]]t^3 + ...$$

Does this look familiar? For differentiable functions f and t close to 0

$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + \frac{1}{3!}f'''(0)t^3 + ...$$

This is a Taylor series expansion of f, which requires taking derivatives of all powers. The above formulas involving Q, D, and H require, then, "derivatives" of transformations. Thank you!