A Covariant Stone-von Neumann Theorem

Joint with: Leonard Huang (U. Nevada, Reno)

Publication: Comm. in Math. Phys., Vol. 378, Issue 1, 117-147

Some History

Open Question: Is every pair of s.a. operators on a Hilbert space which satisfies the HCR equivalent to the Schrödinger pair?

Schrödinger pair

- $P: \text{dom}(P) \to L^2(\mathbb{R})$
  $Pf = -i \frac{df}{dx} \quad \forall f \in S(\mathbb{R})$
- $Q: \text{dom}(Q) \to L^2(\mathbb{R})$
  $(Qf)(x) = x f(x) \quad \forall f \in S(\mathbb{R})$
- $[P, Q] f = if \quad \forall f \in S(\mathbb{R})$

Heisenberg pair

- $A: \text{dom}(A) \to H$ s.a.
- $B: \text{dom}(B) \to H$ s.a
- $\exists K \subseteq \text{dom}(A) \cap \text{dom}(B)$ s.t. $\overline{K} = H$
- $[A, B] k = ik \quad \forall k \in K$

Main Strategy: Determine sufficient condition(s) on K that guarantee "Integrability" of (H, A, B) to (H, R, S) such that

- $R: \mathbb{R} \to \mathcal{U}(H)$, $S: \hat{\mathbb{R}} \to \mathcal{U}(H)$ unitary rep's.
- $S_y R_x = \underbrace{y(x)}_{e^{iyx}} R_x S_y \quad \forall \begin{array}{l} x \in \mathbb{R} \\ y \in \hat{\mathbb{R}} \end{array}$ (Weyl relation)

and then apply classical Stone-von Neumann Theorem

The Classical Stone-von Neumann Theorem

Let G be a l.c.a. group.

<u>Theorem</u>: Suppose $R: G \longrightarrow \mathcal{U}(H)$ and $S: \hat{G} \longrightarrow \mathcal{U}(H)$ are unitary gp. representations such that

$$S_\gamma R_x = \gamma(x) R_x S_\gamma \quad \forall \gamma \in \hat{G},\ x \in G.$$

Then $\exists\ W: H \rightarrow \oplus L^2(G)$ s.t. $R \sim_W \oplus U$ and $S \sim_W \oplus V$

where for each $f \in C_0(G)$,

$$\forall x \in G: \ (U_x f)(y) = f(x^{-1}y) \qquad \forall y \in G$$
$$\forall \gamma \in \hat{G}: \ (V_\gamma f)(y) = \gamma(y) f(y) \qquad \forall y \in G$$

<u>Def'n</u>: (H, R, S) is a G-Heisenberg rep'n

<u>Def'n</u>: $(L^2(G), U, V)$ is <u>the</u> G-Schrödinger rep'n

<u>Corollary</u>: If (H, A, B) is a Heisenberg pair that integrates to (H, R, S), a Heisenberg rep. then,

$$(H, A, B) \sim_W \oplus \left(L^2(G), P, Q\right).$$

Our Work

Goal 1: $\underbrace{\text{Extend S-vN Theorem}}$ to unit. gp. rep'ns on Hilbert C^*-module setting.

* **Goal 2:** " to covariant rep'n of C^*-dynamical systems & unitary gp. representations.

Goal 3: Determine an integrability criterion for pairs of s.a. operators on a Hilbert C^*-module.

Throughout, (A, G, α) is a C^*-dynamical system w/ G l.c.a.

Theorem (Huang-Ismert, 2020)
Every $(\mathbb{K}(H), \mathbb{R}, \alpha)$-Heisenberg representation is unitarily equivalent to a direct sum of copies of the $(\mathbb{K}(H), \mathbb{R}, \alpha)$-Schrödinger rep'n.

(A, G, α) - Representations

Def'n: An (A, G, α)-Heisenberg representation is a quadruple (X, ρ, r, s), where

- X is a full Hilbert A-module
- $\rho : A \to \mathcal{L}(X)$ nondeg. $*$-rep.
- $r : G \to \mathcal{U}(X)$ and $s : \hat{G} \to \mathcal{U}(X)$ strongly-cts. unitary rep-s.
- $s_\gamma r_x = \gamma(x) r_x s_\gamma \quad \forall x \in G, \gamma \in \hat{G}$
- (ρ, r) is a covariant pair for (A, G, α)
- (ρ, s) is a covariant pair for $(A, \hat{G}, 1)$

Def'n: The (A, G, α)-Schrödinger representation is the quadruple $(L^2(A, G, \alpha), M, u, v)$, where:

$L^2(A, G, \alpha)$ is the completion of $C_c(G, A)$ as a right A-module where $\forall f, g \in C_c(G, A)$, $\forall a \in A$

- $(f \cdot a)(x) = f(x) \alpha_x(a)$

$$\langle f | g \rangle = \int_G \alpha_{-x}\left(f(x)^* g(x)\right) d\mu(x)$$

- $M : A \to \mathcal{L}(L^2(A, G, \alpha))$

$$M(a) f = af \qquad \forall f \in C_c(G, A)$$

- $u : G \to \mathcal{U}(L^2(A, G, \alpha))$, $v : \hat{G} \to \mathcal{U}(L^2(A, G, \alpha))$

$$(u_x f)(y) = \alpha_x(f(x^{-1} y)) \quad \text{and} \quad (v_\gamma f)(y) = \gamma(y) f(y)$$

Proposition: The (A, G, α)-Schrödinger representation is a (A, G, α)-Heisenberg representation.

Classical to Covariant, Step 1

Theorem: $C_0(G) \rtimes_{lt} G \xrightarrow{\cong} K(L^2(G))$ via $M \rtimes U$, where

$$M: C_0(G) \to B(L^2(G)) \qquad f \mapsto M_f$$

$$U: G \to \mathcal{U}(L^2(G)) \qquad (U_x g)(y) = g(x^{-1}y)$$

form a covariant rep'n of $(C_0(G), G, lt)$.

$$K(L^2(G)) \xrightarrow{(M \rtimes U)^{-1}} C_0(G) \rtimes_{lt} G$$

Theorem: $C_0(G,A) \rtimes_{lt \otimes \alpha} G \xrightarrow{\cong} K(L^2(A,G,\alpha))$

Proof (Outline):

- If X is a B-A imprimitivity bimodule, then
  $$\mathcal{B} \cong K(X_A)$$

- Green's Imprimitivity Theorem gives $L^2(A,G,\alpha)$ is a
  $$C_0(G,A) \rtimes_{lt \otimes \alpha} G \,-\, A$$
  imprimitivity bimodule. $\therefore$
- This isomorphism is implemented by $\Xi \rtimes u$
- $\Xi : C_0(G,A) \to \mathcal{L}(L^2(A,G,\alpha)) \qquad f \mapsto \{(\Xi_f g) = fg\}$

- (Ξ, u) covariant pair.

$$\rightsquigarrow K(L^2(A,G,\alpha)) \xrightarrow[\cong]{(\Xi \rtimes u)^{-1}} C_0(G,A) \rtimes_{lt \otimes \alpha} G \longrightarrow \mathcal{L}(X)$$

Classical to Covariant, Step 2

Proposition:

$$\left\{\begin{array}{c}\text{G-Heisenberg representations}\\ (R, S, H)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Covariant rep'n's } (\pi, R)\\ \text{of } (C_0(G), G, \mathrm{lt})\end{array}\right\}$$

$(\longrightarrow)$: $\pi_S : (\iota_{C_0(G)} \rtimes S) \circ \mathcal{F}^{-1}$

- (π_S, R) covariant rep. of $(C_0(G), G, \mathrm{lt})$
- $M = \pi_V$

$$\rightsquigarrow \quad K(L^2(G)) \xrightarrow{(\pi_V \rtimes u)^{-1}} C_0(G) \rtimes_{\mathrm{lt}} G \xrightarrow{\pi_S \rtimes R} B(H)$$

Proposition (Huang-Ismert, 2020)

$$\left\{\begin{array}{c}(A, G, \alpha)\text{-Heisenberg representations}\\ (X, \rho, r, s)\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Covariant rep'n's}\\ (\Pi, r)\\ \text{of } (C_0(G,A), \mathrm{lt} \otimes \alpha, G)\end{array}\right\}$$

$(\longrightarrow)$: $\Pi_{\rho,s} := (\rho \rtimes s) \circ \mathcal{F}_A^{-1}$ where $\mathcal{F}_A : C_c(\hat{G}, A) \to C_0(G, A)$

- $(\Pi_{\rho,s}, r)$ form a covariant rep. for $(C_0(G,A), G, \mathrm{lt} \otimes \alpha)$
- $\Pi_{M,V} = \Xi$

$$\rightsquigarrow \quad K(L^2(A, G, \alpha)) \underset{\simeq}{\xrightarrow{(\cancel{\Xi}^{\,\Pi_{M,V}} \rtimes u)^{-1}}} C_0(G, A) \rtimes_{\mathrm{lt} \otimes \alpha} G \xrightarrow{\Pi_{\rho,s} \rtimes r} \mathcal{L}(X)$$

Classical to Covariant, Step 3

Theorem (Arveson) Every non-degenerate $*$-rep'n of $K(H')$ is unitarily equivalent to (a direct sum of copies of) the identity representation.

$H' = L^2(G)$

$\psi = (\pi_S \rtimes R) \circ (\pi_V \rtimes u)^{-1}$

$$\begin{tikzcd} K(H') \arrow[r, "\psi"] \arrow[d, "\oplus \mathrm{id}"'] & B(H) \arrow[dl, dashed, "\sim W"] \\ \oplus B(H') & \end{tikzcd}$$

$$\begin{tikzcd} K(L^2(G)) \arrow[r, "(\pi_V \rtimes u)^{-1}"] \arrow[d, "\oplus \mathrm{id}"'] & C_0(G) \rtimes_{lt} G \arrow[r, "(\pi_S \rtimes R)"] & B(H) \arrow[dll, dashed, "\sim W"] \\ \oplus B(L^2(G)) & & \end{tikzcd}$$

Theorem (Huang-Ismert, 2020) Every non-degenerate $*$-rep'n of $K(X)$, where X is a Hilbert $K(H)$-module, is unitarily equivalent to (a direct sum of copies of) $\mathrm{id}: K(X) \to K(X)$.

$$\begin{tikzcd} K(L^2(A,G,\alpha)) \arrow[r, "(\Pi_{M_V} \rtimes u)^{-1}", "\cong"'] \arrow[d, "\oplus \mathrm{id}"'] & C_0(G,A) \rtimes_{lt \otimes \alpha} G \arrow[r, "(\Pi_{P,S} \rtimes r)"] & \mathcal{L}(X) \arrow[dll, dashed, "\sim W"] \\ \oplus \mathcal{L}(L^2(A,G,\alpha)) & & \end{tikzcd}$$

IF
$A = \mathbb{K}(H)$
$G = \mathbb{R}$

Conclusion

There's a unitary $W : X \to \oplus L^2(A, G, \alpha)$ such that

$$(\Pi_{\rho,s} \rtimes r) \circ (\Pi_{M,V} \rtimes u)^{-1} \sim_W \oplus \, id_{K(L^2(A,G,\alpha))}$$

$$\Rightarrow \quad \Pi_{\rho,s} \rtimes r \sim_W \oplus \; \Pi_{M,V} \rtimes u$$

$$\Rightarrow \quad \Pi_{\rho,s} \sim_W \oplus \, \Pi_{M,V} \quad \text{and} \quad r \sim_W \oplus \, u$$

$$\Rightarrow \quad \rho \sim_W \oplus M, \quad s \sim_W \oplus V, \quad \text{and} \quad r \sim_W \oplus \, u$$

$\boxtimes$

Future Directions

- Coming soon: Uniqueness statement for pairs of s.a. operators that form the appropriate analogue of a Heisenberg pair.
- More general A
- Non-abelian G - co-actions?
- Quantum gps.?