Quantum graphs and their infinite path spaces

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What operator algebraists do

Let $G = (V, A)$ be a finite simple graph with edge set E and infinite path space X_A:

$$\Rightarrow \quad \mathcal{O}_A, \quad \mathcal{O}_E, \quad C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

In fact, $\mathcal{O}_A \cong \mathcal{O}_E \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.

Let $\mathcal{G}$ be a *quantum graph*.

- What do associated C^*-algebras look like?
- What is their relationship?

A classical graph's Cuntz–Krieger algebra

Let $G = (V, A)$ be a finite simple graph with $|V| = n$.

Definition

A Cuntz–Krieger (CK) G-family is a set $\{S_i : 1 \leq i \leq n\}$ in a C^*-algebra D such that for each $1 \leq i \leq n$

- S_i is a partial isometry
- $S_i^* S_i = \sum_j A_{ij} S_j S_j^*$
- $\sum_i S_i S_i^* = 1$

The Cuntz–Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by a CK G-family.

A classical graph's Cuntz–Pimsner algebra

Let $G = (V, A)$ be a finite simple graph. The edge set E for G is

$$E = \{(v, w) : A_{vw} = 1\} \subseteq V \times V \quad \text{(read from right to left).}$$

Definition

The *edge correspondence* for G is the C^*-correspondence $C(E) \subseteq C(V \times V)$ over $C(V)$ where for any $\xi, \eta \in C(E)$, $f \in C(V)$, $(v, w) \in E$:

- $(\xi \cdot f)(v, w) := \xi(v, w) f(v)$
- $(f \cdot \xi)(v, w) := f(w) \xi(v, w)$
- $\langle \xi, \eta \rangle (v) = \sum_{v \leftarrow w} \overline{\xi(v, w)} \eta(v, w)$

Can construct the Cuntz–Pimsner algebra $\mathcal{O}_E$, which is universal with respect to covariant Toeplitz representations of $C(E)$.

A classical graph's Exel crossed product

Let $G = (V, A)$ be a finite simple graph. Define

$$X_A := \{(v_{k_1}, v_{k_2}, ...) : v_{k_i} \in V,\ A_{k_i k_{i+1}} = 1\ \forall i \in \mathbb{N}\}.$$

Consider the left shift $\sigma : X_A \to X_A$ given by

$$\sigma(v_{k_1}, v_{k_2}, ...) := (v_{k_2}, v_{k_3}, ...).$$

Definition

Suppose G has no sinks. The natural Exel system associated to G is the triple $(C(X_A), \alpha, \mathcal{L})$ where α, $\mathcal{L} : C(X_A) \to C(X_A)$ act on $f \in C(X_A)$ by:

- $\alpha(f) := f \circ \sigma$
- $[\mathcal{L}(f)](\theta) := \frac{1}{|\sigma^{-1}(\{\theta\})|} \sum_{\gamma \in \sigma^{-1}(\{\theta\})} f(\gamma) \quad \forall \theta \in X_A$

$(C(X_A), \alpha, \mathcal{L})$ gives rise to the Exel crossed product $C(X_A) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$.

Summary of C^*-algebras associated to classical graphs

Graph structure	C^*-algebra
A	$\mathcal{O}_A$
$C(E)$	$\mathcal{O}_E$
(X_A, σ)	$C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

Theorem

If G has no sinks and no sources, then

$$\mathcal{O}_A \cong \mathcal{O}_E \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}.$$

Summary of C^*-algebras associated to classical graphs

Classical graphs

Structure	C^*-algebra
A	$\mathcal{O}_A$
$C(E)$	$\mathcal{O}_E$
(X_A, σ)	$C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

$$\mathcal{O}_A \cong \mathcal{O}_E \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

Quantum graphs

Structure	C^*-algebra

Quantum sets

Definition

A *quantum set* is a pair (B, ψ) where

- B is a (finite-dimensional) C^*-algebra
- ψ is a δ-form:

$$B \xrightarrow{m^*} B \otimes B \xrightarrow{m} B \equiv B \xrightarrow{\delta^2 \mathrm{id}} B.$$

Example

Let V be a finite set of order $n \Rightarrow C(V) \cong \mathbb{C}^n$.
Let $\{p_v : v \in V\}$ be the basis for $C(V)$, and define

$$\psi(p_v) := \frac{1}{n} \quad \forall v \in V.$$

Then $(C(V), \psi)$ is a (commutative) quantum set with $\delta^2 = n$.

Quantum sets (vertices)

Let $B = \bigoplus_{a=1}^{d} M_{N_a}(\mathbb{C})$ with basis $\{e_{ij}^{(a)} : 1 \leq a \leq d, 1 \leq i, j \leq N_a\}$.

Example

Define $\psi : B \to \mathbb{C}$ by

$$\psi(x) = \frac{1}{\dim B} \sum_{a=1}^{d} N_a \mathrm{Tr}(x^{(a)}) \quad \forall x \in M_{N(a)}(\mathbb{C}),$$

a.k.a., "the right trace." $\Rightarrow (B, \psi)$ is a quantum set w/ $\delta^2 = \dim B$.

Example

If $q \in B$ is a density matrix, and each $q^{(a)} \in M_{N_a}(\mathbb{C})$ is invertible with $\mathrm{Tr}((q^{(a)})^{-1}) = \delta^2$, then $\psi(x) := \sum_{a=1}^{d} \mathrm{Tr}(q^{(a)} x)$ defines a δ-form on B with $\delta^2 = \dim B$.

Quantum graphs

Definition

A *quantum graph* is a triple (B, ψ, A) where (B, ψ) is a quantum set and A is a quantum adjacency matrix for (B, ψ):

$$B \xrightarrow{m^*} B \otimes B \xrightarrow{A \otimes A} B \otimes B \xrightarrow{m} B \quad \equiv \quad B \xrightarrow{\delta^2 A} B$$

Example

Let $G = (V, A)$ be a finite simple graph.
$\Rightarrow (C(V), \frac{1}{|V|}, A)$ is a quantum graph w/ $\delta^2 = n^2$.

Remark: All finite simple graphs can be viewed as quantum graphs.

Quantum graphs

Let (B, ψ) be a finite quantum set.

Example (Complete quantum graph)

Define $A : B \to B$ on $x \in B$ by

$$A(x) = \delta^2 \mathrm{Tr}(x) 1_B.$$

Notation: $K(B, \psi)$

Example (Trivial quantum graph)

Define $A : B \to B$ on $x \in B$ by

$$A(x) = x.$$

Notation: $T(B, \psi)$

Quantum Cuntz–Krieger algebras

Let $\mathcal{G} := (B, \psi, A)$ be a quantum graph with $B = \bigoplus_{a=1}^{d} M_{N_a}(\mathbb{C})$.

Definition (BEVW, 2021)

A QCK $\mathcal{G}$-family is a linear map $S : B \to D$ giving rise to a family of operator-valued matrices $\{S^{(a)} : 1 \leq a \leq d\}$ such that for each $1 \leq a \leq d$:

- $S^{(a)}$ with entries $S(e_{ij}^{(a)})$ is a matrix partial isometry
- $S^{(a)*}S^{(a)} = \sum_b A_a^b S^{(b)} S^{(b)*}$
- (BHINW, 2022) $\sum_{a,i,j} S_{ij}^{(a)}(S_{ij}^{(a)})^* = \frac{1}{\delta^2}1$

The QCK algebra $\mathcal{O}(\mathcal{G})$ is the universal C^*-algebra generated by the range of such an S.

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- $S_{ij}^{(a)} = \sum_{k,\ell} S_{ik}^{(a)} (S_{\ell k}^{(a)})^* S_{\ell j}^{(a)} \quad \forall 1 \leq i, j \leq N_a$
- $S^{(a)*} S^{(a)} = \sum_{b} A_a^b S^{(b)} S^{(b)*}$
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- $S_{ij}^{(a)} = \sum_{k,\ell} S_{ik}^{(a)} (S_{\ell k}^{(a)})^* S_{\ell j}^{(a)} \quad \forall 1 \leq i,j \leq N_a$
- $S^{(a)^*} S^{(a)} = \sum_{b} A_a^b S^{(b)} S^{(b)^*}$
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- $S_{ij}^{(a)} = \sum_{k,\ell} S_{ik}^{(a)} (S_{\ell k}^{(a)})^* S_{\ell j}^{(a)} \quad \forall 1 \leq i, j \leq N_a$
- $\sum_{k} (S_{ki}^{(a)})^* S_{kj}^{(a)} = \sum_{b,\ell,m} A_{ija}^{\ell mb} \sum_{n} S_{\ell n}^{(b)} (S_{mn}^{(b)})^* \quad \forall 1 \leq i, j \leq N_a$
- (BHINW, 2022) $\sum_{a,i,j} S_{ij}^{(a)} S_{ij}^{(a)^*} = \delta^2 1$

The QCK algebra $\mathcal{O}(\mathcal{G})$ is the universal C^*-algebra generated by the range of such an S.

Quantum Cuntz–Krieger algebras

Definition

A QCK $\mathcal{G}$-family is a linear map $S : B \to D$ giving rise to a family of operator-valued matrices $\{S^{(a)} : 1 \leq a \leq d\}$ with entries $S_{ij}^{(a)} := S(e_{ij}^{(a)})$ such that for each $1 \leq a \leq d$:

- $S_{ij}^{(a)} = \sum_{k,\ell} S_{ik}^{(a)} (S_{\ell k}^{(a)})^* S_{\ell j}^{(a)} \quad \forall 1 \leq i, j \leq N_a$
- $\sum_{k} (S_{ki}^{(a)})^* S_{kj}^{(a)} = \sum_{b,\ell,m} A_{ija}^{\ell m b} \sum_{n} S_{\ell n}^{(b)} (S_{mn}^{(b)})^* \quad \forall 1 \leq i, j \leq N_a$
- $\sum_{a,i,j} S_{ij}^{(a)} {S_{ij}^{(a)}}^* = \delta^2 1$

The QCK algebra $\mathcal{O}(\mathcal{G})$ is the universal C^*-algebra generated by the range of such an S, $\{S(e_{ij}^{(a)}) : 1 \leq a \leq d, 1 \leq i, j \leq N_a\}$.
Remark: these relations are very “coarse"—They do not (as far as we know) guarantee generators to be partial isometries.

Summary of C^*-algebras associated to (quantum) graphs

Let $G = (V, A)$ be a classical (simple) graph and $\mathcal{G} = (B, \psi, A)$ be a quantum graph.

Classical

Structure	C^*-algebra
A	$\mathcal{O}_A$
$C(E)$	$\mathcal{O}_E$
(X_A, σ)	$C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

$\mathcal{O}_A \cong \mathcal{O}_E \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

Quantum

Structure	C^*-algebra
A	$\mathcal{O}(\mathcal{G})$

Summary of C^*-algebras associated to (quantum) graphs

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$$\mathcal{O}_A \cong \mathcal{O}_E \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

Quantum

Structure	C^*-algebra
A	$\mathcal{O}(\mathcal{G})$
?	?

Quantum edge correspondences

Classically, $C(E) \subseteq C(V \times V) \cong C(V) \otimes C(V)$, and $C(E)$ is generated as C^*-correspondence over $C(V)$ by

$$1_E = \sum_{e \in E} \xi_e = \sum_{(v,w) \in V \times V} A_{vw} \xi_{(v,w)} = \sum_{(v,w) \in V \times V} A_{vw} p_v \otimes p_w.$$

Definition (BHINW, 2022)

Let $\mathcal{G} := (B, \psi, A)$ be a quantum graph. The quantum edge correspondence E for $\mathcal{G}$ is the C^*-correspondence

$$\operatorname{span}\{x \cdot \varepsilon \cdot y : x, y \in B\} \subseteq B \otimes_\psi B$$

over B, where $\varepsilon := \frac{1}{\delta^2}(\mathrm{id} \otimes A)m^*(1_B)$. If $\mathcal{G}$ is classical, $\varepsilon = 1_E$.

Examples of quantum edge correspondences

Example

Let $K(B,\psi)$ be a complete quantum graph. Then $\varepsilon = 1_B \otimes 1_B$, so

$$E = \text{span}\{x \cdot 1_B \otimes 1_B \cdot y : x, y \in B\} = \text{span}\{x \otimes y : x, y \in B\}$$

is all of $B \otimes_\psi B$.

Example

Let $T(B,\psi)$ be a trivial quantum graph. Then $\varepsilon = \frac{1}{\delta^2} m^*(1_B)$, so

$$E = \text{span}\{x \cdot m^*(1_B) \cdot y : x, y \in B\} = m^*(B),$$

which is isomorphic to B as B-correspondences.

Summary of C^*-algebras associated to (quantum) graphs

Classical

Structure	C^*-algebra
A	$\mathcal{O}_A$
$C(E)$	$\mathcal{O}_E$
(X_A, σ)	$C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

$$\mathcal{O}_A \cong \mathcal{O}_E \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

Quantum

Structure	C^*-algebra
A	$\mathcal{O}(\mathcal{G})$
E	$\mathcal{O}_E$

$$\mathcal{O}(\mathcal{G}) \cong \mathcal{O}_E?$$

Local quantum Cuntz–Krieger algebras

Question: If $\mathcal{G}$ is a quantum graph with quantum edge correspondence E, is it true that $\mathcal{O}(\mathcal{G}) \cong \mathcal{O}_E$?

Answer: We're not sure. But we do know...

Theorem (BHINW, 2022)

Let $\mathcal{G} := (B, \psi, A)$ be a quantum graph and let E be its quantum edge correspondence. Define $J(\mathcal{G})$ to be the ideal in $\mathcal{O}(\mathcal{G})$ generated by the "local relations"

- $\sum_k S_{ik}^{(a)} (S_{\ell k}^{(a)})^* S_{mj}^{(b)} - \delta_{a=b} \delta_{\ell=m} S_{ij}^{(a)}$
- $(S_{ri}^{(a)})^* S_{tj}^{(b)} - \delta_{a=b} \delta_{r=t} \sum_{c,\ell,m} A_{ija}^{\ell mc} \sum_n S_{\ell n}^{(c)} (S_{mn}^{(c)})^*$

Then

$$\mathcal{O}(\mathcal{G}) / J(\mathcal{G}) \cong \mathcal{O}_E.$$

Summary of C^*-algebras associated to (quantum) graphs

Classical

Structure	C^*-algebra
A	$\mathcal{O}_A$
$C(E)$	$\mathcal{O}_E$
(X_A, σ)	$C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

$$\mathcal{O}_A \cong \mathcal{O}_E \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

Quantum

Structure	C^*-algebra
A	$\mathcal{O}(\mathcal{G})$
E	$\mathcal{O}_E$
?	$? \rtimes_{?,?} \mathbb{N}$

$$\mathcal{O}(\mathcal{G})/J(\mathcal{G}) \cong \mathcal{O}_E$$

Exel systems for some quantum graphs

Example (Complete quantum graph)

The quantum edge correspondence E for $K(B, \psi)$ was $B \otimes_\psi B$, "all possible edges."

$\Rightarrow$ The infinite path space should contain "all possible paths."

Consider the C^*-algebra $B^{\otimes \mathbb{N}}$ with $\alpha, \mathcal{L} : B^{\otimes \mathbb{N}} \to B^{\otimes \mathbb{N}}$ by

- $\alpha(f) = 1_B \otimes f$
- $\mathcal{L}(f_1 \otimes f_2 \otimes \ldots) = \psi(f_1) f_2 \otimes f_3 \otimes \ldots$

$\Rightarrow (B^{\otimes \mathbb{N}}, \alpha, \mathcal{L})$ is an Exel system.

Theorem (Brannan-Hamidi-I-Nelson-Wasilewski, 2022)

$$B^{\otimes \mathbb{N}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \cong \mathcal{O}_{\dim B} \cong \mathcal{O}(E).$$

Exel systems for some quantum graphs

Example (Trivial quantum graph)

Recall that the quantum edge correspondence E for $T(B, \psi)$ was $m^*(B) \cong B$, "a loop at each vertex."

$\Rightarrow$ The infinite path space should just be "infinite loops at each vertex." Consider the C^*-algebra B with $\alpha, \mathcal{L} : B \to B$ by

- $\alpha(f) = f$
- $\mathcal{L}(f) = f$

$\Rightarrow$ $(B, \mathrm{id}, \mathrm{id})$ is an Exel system.

Theorem (BHINW + Others)

$$B \rtimes_{id,id} \mathbb{N} \cong B \otimes C(\mathbb{T}) \cong \mathcal{O}(E).$$

Infinite path space for a quantum graph

In examples, the C^*-algebras on which we defined dynamics came from $E^{\otimes_B \mathbb{N}}$:

- Complete: $E = B \otimes_\psi B$

$$\Rightarrow E^{\otimes_B \mathbb{N}} = (B \otimes_\psi B) \otimes_B (B \otimes_\psi B) \ldots \cong B^{\otimes \mathbb{N}}.$$

- Trivial: $E \cong B$

$$\Rightarrow E^{\otimes_B \mathbb{N}} = B \otimes_B B \otimes_B B \ldots \cong B.$$

- General: E is a C^*-correspondence over B

$$\Rightarrow E^{\otimes_B \mathbb{N}} = \text{ a } C^*\text{-correspondence, not a } C^*\text{-algebra}$$

If $E^{\otimes_B \mathbb{N}}$ is not a C^*-algebra, we don't have the ability to construct an Exel system, at least not the existing kind.

Goals (& GOALS)

Classical

Structure	C^*-algebra
A	$\mathcal{O}_A$
$C(E)$	$\mathcal{O}_E$
(X_A, σ)	$C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

$$\mathcal{O}_A \cong \mathcal{O}_E \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

Quantum

Structure	C^*-algebra
A	$\mathcal{O}(\mathcal{G})$
E	$\mathcal{O}_E$
$(C(X_A), \alpha, \mathcal{L})$	$C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

$$\mathcal{O}(\mathcal{G})/J(\mathcal{G}) \cong \mathcal{O}_E$$
$$\cong \; C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$