Quantum Edge Correspondences

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A classical graph's edge correspondence

Let $G = (V, A)$ be a finite simple graph. The edge set E for G is

$$E = \{(v, w) : A_{vw} = 1\} \subseteq V \times V \quad \text{(read from right to left).}$$

Definition

The *edge correspondence* for G is the C^*-correspondence $C(E) \subseteq C(V \times V)$ over $C(V)$ where for any $\xi, \eta \in C(E)$, $f \in C(V)$, $(v, w) \in E$:

- $(\xi \cdot f)(v, w) := \xi(v, w) f(v)$
- $(f \cdot \xi)(v, w) := f(w) \xi(v, w)$
- $\langle \xi, \eta \rangle (v) = \sum_{v \leftarrow w} \overline{\xi(v, w)} \eta(v, w)$

Can construct the Cuntz–Pimsner algebra $\mathcal{O}_E$, which is universal with respect to covariant Toeplitz representations of $C(E)$.

Quantum graphs

In a quantum graph...

Definition

a finite vertex set is replaced by a *finite quantum set* consisting of

- a finite-dimensional C^*-algebra B
- a state ψ that is a δ-form

Definition

The adjacency matrix is replaced by a *quantum adjacency matrix*, a linear map $A : B \to B$ which is quantum Schur idempotent:

$$m(A \otimes A)m^* = \delta^2 A$$

Example

Let $G = (V, A)$ be a finite simple graph.
$\rightsquigarrow (C(V), \frac{1}{|V|}, A)$ is a quantum graph w/ $\delta^2 = n^2$.

Quantum edge correspondences

Consider the generator 1_E of $C(E)$ inside $C(V) \otimes C(V)$ as a $C(V)$-correspondence:

$$1_E = \sum_{e \in E} \xi_e = \sum_{(v,w) \in V \times V} A_{vw} \xi_{(v,w)} = \sum_{(v,w) \in V \times V} p_v \otimes A_{vw} p_w.$$

Definition (BHINW, 2022)

Let $\mathcal{G} := (B, \psi, A)$ be a quantum graph. The quantum edge correspondence E for $\mathcal{G}$ is the C^*-correspondence generated by $\varepsilon := \frac{1}{\delta^2}(\mathrm{id} \otimes A)m^*(1_B)$ over B:

$$\mathrm{span}\{x \cdot \varepsilon \cdot y : x, y \in B\} \subseteq B \otimes_\psi B.$$

Examples of quantum edge correspondences

Let (B, ψ) be a finite quantum set. Let $A : B \to B$ be given by

Example (Complete quantum graph)

$A(x) = \delta^2 \psi(x) 1_B$. Then $\varepsilon = \frac{1}{\delta^2}(\mathrm{id} \otimes A)m^*(1_B) = 1_B \otimes 1_B$, so

$$E = \mathrm{span}\{x \cdot 1_B \otimes 1_B \cdot y : x, y \in B\} = \mathrm{span}\{x \otimes y : x, y \in B\}$$

is all of $B \otimes_\psi B$. Makes sense!

Example (Trivial quantum graph)

$A(x) = x$. Then $\varepsilon = \frac{1}{\delta^2}(\mathrm{id} \otimes A)m^*(1_B) = \frac{1}{\delta^2} m^*(1_B)$, so

$$E = \mathrm{span}\{x \cdot m^*(1_B) \cdot y : x, y \in B\} = m^*(B),$$

which is isomorphic to B as B-correspondences. Makes sense!

Faithfulness and fullness of edge correspondences

Given a C^*-correspondence X over a C^*-algebra $\mathcal{A}$,

- X is *faithful* if $a \cdot x = 0 \; \forall x \in X \Rightarrow a = 0$
- X is *full* if $\overline{\text{span}}\{\langle \xi, \eta \rangle_{\mathcal{A}} : \xi, \eta \in X\} = \mathcal{A}$.

Example (Classical edge correspondence)

Let $G = (V, A)$ be a finite simple graph. Recall following structure: for any $\xi, \eta \in C(E)$, $f \in C(V)$, $(v, w) \in E$,

1. $(\xi \cdot f)(v, w) := \xi(v, w) f(v)$
2. $(f \cdot \xi)(v, w) := f(w) \xi(v, w)$
3. $\langle \xi, \eta \rangle (v) = \sum_{v \leftarrow w} \overline{\xi(v, w)} \eta(v, w)$

Fact:

- If $w \in V$ is a sink, then $\forall v \in V$, $(v, w) \notin E$. $\Rightarrow$ $p_w \cdot \xi = 0 \, \forall \xi \in E$.
- If $v \in V$ is a source, then $\forall \xi, \eta \in C(E)$, sum in item 3 is $0 \Rightarrow$ $p_v \notin \langle C(E), C(E) \rangle_{C(V)}$.

Properties of C^*-correspondences

Let (B, ψ, A) be a quantum graph and E its edge correspondence.

Theorem

Given $x, y, z, w \in B$, we have the following helpful formula:

$$\langle x \cdot \epsilon \cdot y, w \cdot \epsilon \cdot z \rangle_B = \frac{1}{\delta^2} y^* A(x^* w) z.$$

Theorem

1. *E is faithful if and only if has no quantum sources, i.e., no central summand of B belongs to $\ker(A)$.*
2. *E is full if and only if $\mathcal{G}$ has no quantum sinks, i.e., no central summand of B is $\perp$ to $A(B)$.*

Cuntz–Pimsner algebras of quantum edge correspondences

If $G = (V, A)$ is a finite simple graph with edge correspondence $C(E)$ with no sinks, its Cuntz–Krieger algebra $\mathcal{O}_A$ is isomorphic to the Cuntz–Pimsner algebra $\mathcal{O}_{C(E)}$.

Question: If $\mathcal{G}$ is a quantum graph with quantum edge correspondence E, when is it true that the quantum Cuntz–Krieger algebra $\mathcal{O}(\mathcal{G})$ is isomorphic to the Cunz–Pimsner algebra $\mathcal{O}_E$?

Theorem (BHINW, 2022)

Let $\mathcal{G}$ be a quantum graph with no quantum sources and let E be its quantum edge correspondence. Define $J(\mathcal{G})$ to be the ideal in $\mathcal{O}(\mathcal{G})$ generated by "local QCK relations." Then

$$\mathcal{O}(\mathcal{G})/J(\mathcal{G}) \cong \mathcal{O}_E.$$

Thank you!