Quantum graphs and their infinite path spaces

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Joint Math Meetings, Boston, MA
MSRI Special Session on Dynamics and Operator Algebras

January 6, 2023

A story about classical graphs

Let $G = (V, A)$ be a finite simple graph.

- $\{0,1\}$-matrix $A \rightsquigarrow$ Cuntz–Krieger algebra $\mathcal{O}_A$
- edge correspondence $C(E)$ (over $C(V)$) $\rightsquigarrow \mathcal{O}_{C(E)}$
- Exel system $(C(E^\infty), \alpha, \mathcal{L}) \rightsquigarrow C(E^\infty) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

“Fact": $\mathcal{O}_A \cong \mathcal{O}_{C(E)} \cong C(X_A) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.

A classical graph's Exel crossed product

Let $G = (V, A)$ be a finite simple graph. Define

$$E^{\infty} := \{(v_{k_1}, v_{k_2}, \ldots) : v_{k_i} \in V,\ A_{k_i k_{i+1}} = 1\ \forall i \in \mathbb{N}\}.$$

Consider the left shift $\sigma : E^{\infty} \to E^{\infty}$ given by

$$\sigma(v_{k_1}, v_{k_2}, \ldots) := (v_{k_2}, v_{k_3}, \ldots).$$

Definition

If G has no sinks, define α, $\mathcal{L} : C(E^{\infty}) \to C(E^{\infty})$ by:

- $\alpha(f) := f \circ \sigma$
- $[\mathcal{L}(f)](\theta) := \frac{1}{|\sigma^{-1}(\{\theta\})|} \sum_{\gamma \in \sigma^{-1}(\{\theta\})} f(\gamma) \quad \forall \theta \in E^{\infty}$

$(C(E^{\infty}), \alpha, \mathcal{L}) \quad \rightsquigarrow \quad$ Exel crossed product $C(E^{\infty}) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$.

Quantum graphs

In a quantum graph...

Definition

a finite vertex set is replaced by a *quantum set* consisting of

- a (finite-dimensional) C^*-algebra B
- a state ψ that is a δ-form

Definition

The adjacency matrix is replaced by a *quantum adjacency matrix*, a linear map $A : B \to B$ which is quantum idempotent:

$$m(A \otimes A)m^* = \delta^2 A$$

Quantum graphs

Example

Let (V, A) be a finite graph. Then $B := C(V)$ is a f.d. C^*-algebra. Let $\{p_v : v \in V\}$ be the basis for $C(V)$, and define

$$\psi(p_v) := \frac{1}{|V|} \quad \forall v \in V.$$

$(C(V), \psi)$ is a quantum set and $(C(V), \psi, A)$ is a quantum graph.

Remark: All finite simple graphs can be viewed as quantum graphs.
Note: $C(E^2) \cong C(E) \otimes_{C(V)} C(E)$, and $C(E^\infty) \cong C(E)^{\otimes_{C(V)}\mathbb{N}}$

Quantum graphs

Let (B, ψ) be a finite quantum set.

Example (Complete quantum graph)

Define $A : B \to B$ on $x \in B$ by

$$A(x) = \delta^2 \psi(x) 1_B.$$

Notation: $K(B, \psi)$

Edges and Paths

- $E \cong B \otimes_\psi B$ ("all possible pairs of vertices")
- $E^2 \cong E \otimes_B E \cong (B \otimes_\psi B) \otimes_B (B \otimes_\psi B) \cong B \otimes_\psi B \otimes_\psi B$

Quantum graphs

Let (B, ψ) be a finite quantum set.

Example (Trivial quantum graph)

Define $A : B \to B$ on $x \in B$ by

$$A(x) = x.$$

Notation: $T(B, \psi)$

Edges and Paths

- $E \cong B$ (“just a loop at each vertix")
- $E^2 \cong E \otimes_B E \cong B \otimes_B B \cong B$

Let $\mathcal{G}$ be a *quantum graph*.

- BEVW '21 and BHINW '22 defined a quantum Cuntz–Krieger algebra $\mathcal{O}(\mathcal{G})$
- BHINW '22 defined a quantum analogue of $C(E)$, from which we can build a Cuntz–Pimsner algebra.
- We have examples of a reasonable is the infinite path space for a quantum graph?
- Is there a natural Exel system whose crossed product recovers the quantum graph's Cuntz–Krieger algebra?

Exel systems for some quantum graphs

Example (Complete quantum graph)

The quantum edge correspondence is $E \cong B \otimes_\psi B$

$\rightsquigarrow$ quantum infinite path space should contain "all possible paths."

Consider $E^{\otimes_B \mathbb{N}} = (B \otimes_\psi B) \otimes_B (B \otimes_\psi B) \ldots \cong B^{\otimes \mathbb{N}}$ with $\alpha, \mathcal{L} : B^{\otimes \mathbb{N}} \to B^{\otimes \mathbb{N}}$ defined by

- $\alpha(f) = 1_B \otimes f$
- $\mathcal{L}(f_1 \otimes f_2 \otimes ...) = \psi(f_1) f_2 \otimes f_3 \otimes ...$

$(B^{\otimes \mathbb{N}}, \alpha, \mathcal{L})$ is an Exel system.

Theorem (Brannan-Hamidi-I-Nelson-Wasilewski, 2022)

$$B^{\otimes \mathbb{N}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \cong \mathcal{O}_{\dim B} \cong \mathcal{O}_E.$$

Exel systems for some quantum graphs

Example (Trivial quantum graph)

The quantum edge correspondence is $E \cong B$

$\rightsquigarrow$ quantum infinite path space should contain "infinite loops at each vertex."

Consider $E^{\otimes_B \mathbb{N}} = B \otimes_B B \otimes_B B \ldots \cong B$ with $\alpha, \mathcal{L} : B^{\otimes \mathbb{N}} \to B^{\otimes \mathbb{N}}$ defined by

- $\alpha(f) = f$
- $\mathcal{L}(f) = f$

$(B, \mathrm{id}, \mathrm{id})$ is an Exel system.

Theorem (BHINW + Others)

$$B \rtimes_{id,id} \mathbb{N} \cong B \otimes C(\mathbb{T}) \cong \mathcal{O}_E.$$

Infinite path space for a quantum graph

In examples, the C^*-algebras on which we defined dynamics to build an Exel system came from $E^{\otimes_B \mathbb{N}}$. But in general, E is a C^*-correspondence over B, not a C^*-algebra, as is $E^{\otimes_B \mathbb{N}}$.

If $E^{\otimes_B \mathbb{N}}$ is not a C^*-algebra, we can't construct an Exel system in the same way as in the examples.

Thank you!