Distinguishing C^*-algebras related to quantum graphs

Lara Ismert
w/ M. Hamidi and B. Nelson

Great Plains Operator Theory Symposium
University of Nebraska - Lincoln

June 4, 2024

C^*-algebras associated to graphs

Let $G = (V, A)$ be a finite simple graph.

Definition

A *Cuntz–Krieger (CK) G-family* is a representation $S : V \to \mathcal{D}$ satisfying

- $S_i S_i^* S_i = S_i \quad \forall i \in V$
- $S_i^* S_i = \sum_{j \in V} A_{ij} S_j S_j^* \quad \forall i \in V$
- $\sum_{i \in V} S_i S_i^* = 1$

The Cuntz–Krieger algebra $\mathcal{O}_A$ is the C^*-algebra generated by a universal CK G-family.

Theorem (Well-known)

When G is row-finite and has no sources, $\mathcal{O}_A \cong \mathcal{O}_E$, where E is the edge correspondence for G and $\mathcal{O}_E$ is its Cuntz–Pimsner algebra.

The quantum setting

When $\mathcal{G}$ is a *quantum graph*,

- $\mathcal{G}$ has an analogous Cuntz–Krieger algebra, $\mathcal{O}(\mathcal{G})$
- $\mathcal{G}$ has an analogous C^*-correspondence $E_{\mathcal{G}}$

Question: For a quantum graph $\mathcal{G}$, does the analogous isomorphism $\mathcal{O}(\mathcal{G}) \cong \mathcal{O}_{E_{\mathcal{G}}}$ hold? Do we want it to?

Quantum graphs

Recall: Given a finite simple graph $G = (V, A)$, the algebra $C(V)$ is just $\mathbb{C}^{|V|}$ and $A : C(V) \to C(V)$ is a linear Schur (entry-wise) idempotent. $\star$

Definition

A (finite) *quantum set* is a (finite-dimensional) C^*-algebra B equipped with a special type of state ψ. A *quantum Schur idempotent* on (B, ψ) is a linear map $A : B \to B$ such that

$$m(A \otimes A)m^* = (\dim B)A$$

A *quantum graph* is a triple (B, ψ, A). $\star$

Example

Given $G = (V, A)$, let $B := C(V)$ and $\psi(p_v) = \frac{1}{|V|}\ \forall v \in V$. Then $\mathcal{G} = (B, \psi, A)$ is a quantum graph.

Let $\mathcal{G} := (B, \psi, A)$ be a quantum graph.

Definition

A *QCK $\mathcal{G}$-family* is a representation $S : B \to \mathcal{D}$ such that

- $\mu(\mu \otimes \mathrm{id})(S \otimes S^* \otimes S)(m^* \otimes \mathrm{id})m^* = S$
- $\mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A$
- $\mu(S \otimes S^*)m^*(1) = 1$

The *quantum Cuntz–Krieger (QCK) algebra* $\mathcal{O}(\mathcal{G})$ is the C^*-algebra generated by a universal QCK $\mathcal{G}$-family.

When G is a classical graph, its QCK algebra $\mathcal{O}(G)$ is isomorphic to its Cuntz–Krieger algebra $\mathcal{O}_A$!

C^*-correspondences

Definition

A *C^*-correspondence over B* is a right Hilbert B-module E equipped with a left B-action $\varphi : B \to \mathcal{L}(E)$. $\star$

Example

Given $E \subset V \times V$, $C(E)$ is a natural $C(V)$-correspondence as a subspace of $C(V \times V) \cong C(V) \otimes C(V)$, and is generated by $\chi_E \in C(E)$.

Definition

Given a quantum graph $\mathcal{G} = (B, \psi, A)$ s.t. A is cp, define

$$\varepsilon_{\mathcal{G}} := \frac{1}{\dim B}(\mathrm{id} \otimes A)m^*(1) \in B \otimes B$$

and define $E_{\mathcal{G}} := \mathrm{Span}\left(\{x \cdot \varepsilon \cdot y : x, y \in B\}\right) \subset B \otimes B$ to be the *quantum edge correspondence* for $\mathcal{G}$.

Complete quantum graph

Example

Let (B, ψ) be a quantum set and define $A(x) = \dim(B) \cdot \psi(x)1$. We call $K(B, \psi) := (B, \psi, A)$ the *complete quantum graph* on (B, ψ).

- $\varepsilon_K = 1 \otimes 1$
- $E_K = \mathrm{Span}\left(\{x \cdot 1 \otimes 1 \cdot y : x, y \in B\}\right) = B \otimes B$.

Theorem (Brannan-Eifler-Voigt-Weber (2022))

When ψ satisfies a weird property, $\mathcal{O}(K(B, \psi))$ is isomorphic to $\mathcal{O}_{\dim B}$, the Cuntz algebra on $\dim B$ generators.

Brannan-Hamidi-I.-Nelson-Wasilewski (2023): For any (B, ψ), the Cuntz–Pimsner algebra $\mathcal{O}_{E_K}$ is isomorphic to $\mathcal{O}_{\dim B}$.

For certain ψ's, $\mathcal{O}(K(B, \psi)) \cong \mathcal{O}_{E_K}$.

Relative Cuntz–Pimsner algebras

Let E be a C^*-correspondence over a C^*-algebra B.

Definition

A *Toeplitz representation* of E is a pair (π, t) satisfying:

- $\pi : B \to \mathcal{D}$ $*$-homomorphism and $t : E \to \mathcal{D}$ linear
- $t(e)\pi(b) = t(e \cdot b)$
- $\pi(\langle e, f \rangle) = t(e)^* t(f)$

The Toeplitz algebra $\mathcal{T}(E)$ is the C^*-algebra generated by the universal Toeplitz representation.

Note: $\mathcal{T}(E)$ can be concretely constructed! Define $t_E : E \to \mathcal{L}(\mathcal{F}(E))$ by $t_E(e)\xi = e \otimes \xi$ and $\pi_E : B \to \mathcal{L}(\mathcal{F}(E))$ by $\pi_E(b)\xi = b \cdot \xi$ for all $\xi \in \mathcal{F}(E)$.

Relative Cuntz–Pimsner algebras

Let B be a f.d. C^*-algebra and E a f.d. faithful B-correspondence.

For any Toeplitz representation (π, t) of E, there is a $*$-representation $\psi_t : K(E) \to \mathcal{L}(\mathcal{F}(E))$ given by $\psi_t(\theta_{e,f}) := t(e)t(f)^*$.

Definition

(π, t) is *co-isometric on* $K \triangleleft B$ if for all $b \in K$, $\psi_t(\varphi(b)) = \pi(b)$.$\star$

Definition

The *relative Cuntz–Pimsner algebra* $\mathcal{O}(K, E)$ is the C^*-algebra generated by a universal Toeplitz representation which is co-isometric on K.

- $K = \{0\} \Rightarrow \mathcal{O}(K, E_{\mathcal{G}}) = \mathcal{T}(E)$
- $K = B \Rightarrow \mathcal{O}(K, E_{\mathcal{G}}) = \mathcal{O}_{E_{\mathcal{G}}}$, the Cuntz–Pimsner algebra of $E_{\mathcal{G}}$.$\star$

QCK $\mathcal{G}$-families from Toeplitz representations

Let $\mathcal{G} := (B, \psi, A)$ be a quantum graph such that $E_{\mathcal{G}}$ is faithful.

Theorem (Brannan-Hamidi-I.-Nelson-Wasilewski (2023))

Every Toeplitz representation (π, t) of $E_{\mathcal{G}}$ which is co-isometric on B admits a QCK $\mathcal{G}$-family.

Theorem (Hamidi-I.-Nelson (2024))

Every Toeplitz representation (π, t) of $E_{\mathcal{G}}$ which is co-isometric on $K := B \cdot A(B) \cdot B$ admits a QCK $\mathcal{G}$-family.

Corollary

Let $K = B \cdot A(B) \cdot B$. When $E_{\mathcal{G}}$ is faithful, $\mathcal{O}(\mathcal{G})$ surjects onto $\mathcal{O}_{E_{\mathcal{G}}}$ and $\mathcal{O}(K, E_{\mathcal{G}})$, and the surjection of $\mathcal{O}(\mathcal{G})$ onto $\mathcal{O}_{E_{\mathcal{G}}}$ factors through $\mathcal{O}(K, E_{\mathcal{G}})$.

We now have $\mathcal{O}(K, E_{\mathcal{G}})$ "sitting between" $\mathcal{O}_{E_{\mathcal{G}}}$ and $\mathcal{O}(\mathcal{G})$.

Finally, a distinction

Let $\mathcal{G} := (B, \psi, A)$ be a quantum graph such that $E_\mathcal{G}$ is faithful.

- We have $\mathcal{O}(K, E_\mathcal{G})$ "sitting between" $\mathcal{O}_{E_\mathcal{G}}$ and $\mathcal{O}(\mathcal{G})$.
- If $K = B \cdot A(B) \cdot B$ were nontrivial and $\mathcal{O}_{E_\mathcal{G}}$ is simple, then $\mathcal{O}(K, E_\mathcal{G})$ and $O_{E_\mathcal{G}}$ would be distinct.
- In this case, $\mathcal{O}(\mathcal{G})$ would be different from $\mathcal{O}_{E_\mathcal{G}}$.

Example

Details to appear! We constructed a quantum graph $\mathcal{G}$ whose $E_\mathcal{G}$ satisfies:

- $E_\mathcal{G}$ is faithful
- $E_\mathcal{G}$ is not full (so $B \cdot A(B) \cdot B$ is nontrivial)
- $E_\mathcal{G}$ satisfies Condition (S) (Eryzulu et. al. (2022)), so $\mathcal{O}_{E_\mathcal{G}}$ is simple

Corollary

Quantum Cuntz–Krieger algebras are interesting.

Thank you!