Cuntz–Pimsner algebras for quantum graphs

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A *C*-algebra* is a Banach $*$-algebra $\mathcal{A}$ which satisfies the *C*-identity*:

$$\|a^*a\| = \|a\|^2 \quad \forall a \in \mathcal{A}.$$

Broad area of research: Given a base object G, construct a C*-algebra $C^*(G)$. Ideally, (1) $C^*(G)$ remembers G in the sense that:

$$C^*(G) \cong C^*(G') \iff G \sim G'$$

and (2) sub-objects of G correspond with subalgebras of $C^*(G)$.

$G =$ (topological) graph or group, groupoid, dynamical system

$G := (V, E, r, s)$ a discrete graph

A Cuntz–Krieger (CK) G-family in a C*-algebra $\mathcal{D}$ consists of a pair (S, P) of representations $P : V \to \mathcal{D}$ and $S : E \to \mathcal{D}$ such that

- $\{P_v : v \in V\}$ = mutually orthogonal projections
- $\{S_e : e \in E\}$ = partial isometries

satisfying **(G1)** and **(G2)**. A CK G-family (S, P) is universal if for any other (s, p) there is a $*$-hom. $\rho : C^*(S, P) \to C^*(s, p)$ such that

$$\rho(P_v) = p_v, \rho(S_e) = s_e \quad \forall v \in V, e \in E.$$

The graph C*-algebra $C^*(G)$ for G is generated by the universal (S, P).

Theorem

Suppose G is row-finite and every cycle has an entry. Hereditary saturated subsets of V correspond bijectively with ideals of $C^*(G)$.

Cuntz–Krieger algebras

Let $A \in M_d(\{0,1\})$, and define $V = [d]$ with

$$E := \{(i,j) : A_{ji} = 1 \text{ for } i,j \in V\}.$$

Then $G_A = (V, E)$ is a finite simple graph. Let (S, P) be a CK G_A-family in $\mathcal{D}$. **(G1)** and **(G2)** can be rephrased as

- S_e is a partial isometry for all $e \in E$ **(CK1)**
- $S_f^* S_f = \sum_{r(e)=s(f)} S_e S_e^* \quad \forall f \in E$ **(CK2)**
- $\sum_{e \in V} S_e S_e^*$ is a unit for $C^*(S)$. **(CK3)**

We call $S : \mathbb{C}^d \to \mathcal{D}$ a Cuntz–Krieger A-family.

Definition

Let $A \in M_d(\{0,1\})$. The Cuntz–Krieger algebra $\mathcal{O}_A$ is the C*-algebra generated by a universal CK A-family.

When G_A has no sinks and no sources, $C^*(G_A) \cong \mathcal{O}_A$.

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Cuntz–Krieger algebras

Example

Consider $G = (V, A)$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\mathcal{O}_A \cong \mathcal{O}_2$.

Consider the representation $S_e, S_o \in \mathcal{B}(\ell^2(\mathbb{N}))$ given by

$$(x_1, x_2, \dots) \overset{S_e}{\mapsto} (0, x_1, 0, x_2, \dots)$$

$$(x_1, x_2, \dots) \overset{S_o}{\mapsto} (x_1, 0, x_2, 0, \dots)$$

have range projections $S_e S_e^* = P_e$, $S_o S_o^* = P_o$ and source projections $S_e^* S_e = I$, $S_o^* S_o = I$ (CK1) satisfying

- $S_e S_e^* + S_o S_o^* = P_e + P_o = I$ (CK3)
- $S_e^* S_e = S_o^* S_o = I = S_e S_e^* + S_o S_o^*$ (CK2).

Thus, $\{S_e, S_o\}$ is a (universal) CK G-family and $C^*(S_e, S_o)$ is $\mathcal{O}_2$.

Cuntz–Krieger algebras

Generally speaking, Cuntz–Krieger and graph C*-algebras provide an abundance of examples of C*-algebras...

- CK algebras contain all (finitely generated) Cuntz algebras.
- Every $\mathcal{O}_A$ is a graph C*-algebra and an Exel-Laca algebra, and thus an ultragraph C*-algebra. Each is also an étale groupoid C*-algebra.
- Algebraic and topological features of $\mathcal{O}_A$ provide dynamical information about the Markov shift space arising from A.

Quantum graphs are rising objects of interest to the OA community.

1. How do we construct a C*-algebra from a quantum graph $\mathcal{G}$? the (local) quantum Cuntz–Krieger algebra $\mathcal{O}(\mathcal{G})$.
2. What analogous theorems can we prove about $\mathcal{O}(\mathcal{G})$? There are so many to explore–I'll talk about simplicity.

Quantizing Classical Sets

Let $V = [d]$ and consider $C(V) = \mathbb{C}^d = \text{Span}\{p_v : v \in V\}$. Then $C(E)$ is a subspace of $C(V \times V) \cong C(V) \otimes C(V)$. Define

$$m : C(V) \otimes C(V) \to C(V) \quad m(f \otimes g) := f \cdot g.$$

For $T : C(V) \to C(V)$ linear, note $m \circ T \otimes T = T \circ m \iff T$ is a hom.

Example

Equip $C(V)$ with $\psi(p_v) = \frac{1}{d}$ for all v, so

$$m^* : C(V) \to C(V) \otimes C(V) \text{ is given by } m^*(p_v) = d(p_v \otimes p_v).$$

Note $mm^*(f) = df$ for all $f \in C(V)$. We say ψ is a d-form and $(C(V), \psi)$ is a commutative finite quantum set.

Quantum sets

Idea: Make $(C(V), \psi)$ noncommutative.

Definition

A (finite) quantum set is a f.d. C^*-algebra B equipped with a special type of state ψ, called a δ-*form*, which satisfies:

$$\begin{array}{ccc} & & L^2(B \otimes B, \psi \otimes \psi) \\ & \overset{m^*}{\nearrow} & \downarrow m \\ L^2(B, \psi) & \xrightarrow[\delta^2 \, \mathrm{id}_B]{} & L^2(B, \psi) \end{array}$$

Example (Tracial quantum sets)

- When $B = \mathbb{C}^d$, the only δ-form is $\psi(p_v) = \frac{1}{d} \, \forall v \in [d]$.
- When $B = M_n$, the only tracial δ-form is the normalized trace.
- B any f.d. C*-algebra, ψ tracial $\Rightarrow$ normalization s.t. $\delta^2 = \dim B$.

Quantum adjacency matrices

Consider $A \in M_d(\{0,1\})$ as a linear map on $(C(V), \psi)$.TFAE:

- $A \in M_d(\{0,1\})$ is a *linear Schur (entry-wise) idempotent*
- $m \circ A \otimes A \circ m^* = dA$

Definition

Let (B, ψ) be a quantum set. A quantum Schur idempotent on (B, ψ) is a linear map $A : B \to B$ such that

$$m \circ A \otimes A \circ m^* = \delta^2 A.$$

A quantum graph is a triple (B, ψ, A).

Example

Every finite simple graph $G_A = (V, A)$ gives rise to $(C(V), \frac{1}{d}, A)$.

Quantum Cuntz–Krieger algebras

Now we want to quantize the CK relations.
$\mathcal{G} := (B, \psi, A)$ - quantum graph, $\mathcal{D}$ - C*-algebra, $\mu_D : D \otimes D \to D$

Definition

A *QCK $\mathcal{G}$-family* is a linear representation $S : B \to \mathcal{D}$ such that:

- $\mu_D(\mu_D \otimes \mathrm{id})(S \otimes S^* \otimes S)(m^* \otimes \mathrm{id})m^* = S$ (QCK1)
- $\mu_D(S^* \otimes S)m^* = \mu_D(S \otimes S^*)m^* A$ (QCK2)
- $\mu_D(S \otimes S^*)m^*(1) = \delta^{-2}1$ (QCK3)

The quantum Cuntz–Krieger (QCK) algebra $\mathcal{O}(\mathcal{G})$ is the C^*-algebra generated by the universal QCK $\mathcal{G}$-family.

When $G = (V, A)$ is a classical graph, its QCK algebra $\mathcal{O}(G)$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{A^T}$!

QCK algebras for complete quantum graphs

The relations defining $\mathcal{O}(\mathcal{G})$ make it pretty intractable for arbitrary $\mathcal{G}$.

Example (Complete quantum graphs)

Let (B, ψ) be a quantum set and define $A(x) = \delta^2\psi(x)1$. We call $K(B, \psi) := (B, \psi, A)$ the complete quantum graph on (B, ψ).

Theorem (Brannan-Eifler-Voigt-Weber (2022))

When the δ-form ψ satisfies $\delta^2 \in \mathbb{N}$, the QCK algebra $\mathcal{O}(K(B, \psi))$ is isomorphic to the Cuntz algebra on $\dim B$ generators.

Theorem (Brannan-Hamidi-I.-Nelson-Wasilewski (2023))

For any (B, ψ), the local quantum Cuntz–Krieger algebra $L\mathcal{O}(K(B, \psi))$ is isomorphic to $\mathcal{O}_{\dim B}$.

Local quantum Cuntz–Krieger algebras

$\mathcal{G} := (B, \psi, A)$ - quantum graph, $\mathcal{D}$ - C*-algebra, $\mu_D : D \otimes D \to D$

Definition

A local QCK $\mathcal{G}$-family is a linear representation $S : B \to \mathcal{D}$ such that:

- $\mu_D(\mu_D \otimes \mathrm{id})(S \otimes S^* \otimes S)(m^* \otimes \mathrm{id}) = \delta^{-2} Sm$ (LQCK1)
- $\mu_D(S^* \otimes S) = \delta^{-2} \mu_D(S \otimes S^*) m^* A m$ (LQCK2)
- $\mu_D(S \otimes S^*) m^*(1) = \delta^{-2} 1$ (LQCK3)

The *local quantum Cuntz–Krieger (QCK) algebra* $L\mathcal{O}(\mathcal{G})$ is the C^*-algebra generated by the universal LQCK $\mathcal{G}$-family.

The local QCK algebra $L\mathcal{O}(\mathcal{G})$ is a quotient of $\mathcal{O}(\mathcal{G})$. The previous slide says for $K(B, \psi)$ such that $\delta^2 \in \mathbb{N}$, we have $\mathcal{O}(K(B, \psi)) \cong L\mathcal{O}(K(B, \psi))$.

Question: Can we find $\mathcal{G}$ which separates $L\mathcal{O}(\mathcal{G})$ and $\mathcal{O}(\mathcal{G})$? We investigate by simplicity.

Local QCK algebras for single-vertex quantum graphs

Given any quantum set (B, ψ), if A is completely positive, there exist $\{K_i : i \in [p]\} \subset B$ such that $A(x) = \sum_{i \in [p]} K_i^* x K_i$ for all $x \in B$.

Example (Rank-one quantum graphs)

Let (B, ψ) be a quantum set and let $T \in B$ satisfy $\mathrm{Tr}(\rho^{-1} T_a^* T_a) = \delta^2$ for all $a \in [d]$. Then conjugation by T given by $x \mapsto TxT^*$ defines a quantum adjacency matrix on (B, ψ), and we call (B, ψ, ad_T) a rank-one quantum graph.

Theorem (Hamidi-I.-Nelson (2025))

Let $\mathcal{G} := (M_n, \psi, A)$ be a quantum graph such that A is cp. The local QCK algebra $L\mathcal{O}(\mathcal{G})$ is Morita equivalent to the Cuntz algebra on $d(A)$ generators, where $d(A)$ is the dimension of $\mathrm{Span}\{K_i : i \in [p]\}$.

In particular, $L\mathcal{O}(M_n, \psi, A)$ is simple if and only if A is not rank-one.

Edge correspondences

Definition

Let B be a C*-algebra. A *C^*-correspondence over B* is a right Hilbert B-module E equipped with a left B-action $\varphi : B \to \mathcal{L}(E)$.

Example

Let $G = (V, E)$ be a finite classical graph with source and range maps $s, r : E \to V$. Set $C(V) = \text{Span}\,\{p_v : v \in V\}$, $C(E) = \text{Span}\,\{q_e : e \in E\}$.

- $\varphi_E(p_v)(q_e) := p_v(s(e))q_e$
- $q_e \cdot p_v := q_e p_v(r(e))$
- $\langle q_e \mid q_f \rangle_{C(V)}(v) := \sum_{g \in r^{-1}(v)} q_e(g)\overline{q_f(g)}$

Note: $C(E) \subseteq C(V \times V)$ is cyclically generated by χ_E, the "edge checker," as a $C(V)$-bimodule. Denote $C(E)$ with this additional structure by E_G.

Quantum edge correspondences

(B, ψ) - quantum set, define inner product on $B \otimes B$ by

$$\langle x \otimes y \,|\, z \otimes w \rangle := y^* \psi(x^* z) w.$$

Then $B \otimes_\psi B$ is a C*-correspondence generalizing $C(V \times V)$.

Definition

Given a quantum graph $\mathcal{G} = (B, \psi, A)$ s.t. A is cp and ψ is a δ-form, define

$$\varepsilon_{\mathcal{G}} := \frac{1}{\delta^2} (\mathrm{id} \otimes A) m^*(1) \in B \otimes B.$$

The quantum edge correspondence $E_{\mathcal{G}}$ is Span $\{x \cdot \varepsilon \cdot y : x, y \in B\}$.

- When $\mathcal{G}$ is classical, $\varepsilon_{\mathcal{G}}$ recovers χ_E.
- $\epsilon_{\mathcal{G}}$ is the "bridge" to the operator system setting.

A quantum source is a central projection $1_a \in B$ in $\ker(A)$.
A quantum sink is $1_a \in B$ which is orthogonal to $B \cdot A(B) \cdot B$.

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- When $\mathcal{G}$ is classical, $\varepsilon_{\mathcal{G}}$ recovers χ_E.
- $\epsilon_{\mathcal{G}}$ is the "bridge" to the operator system setting.

$\mathcal{G}$ has no quantum sources if left action $\varphi : B \to \mathcal{L}(E_{\mathcal{G}})$ is faithful.
$\mathcal{G}$ has no quantum sinks if $\langle E_{\mathcal{G}} \mid E_{\mathcal{G}} \rangle = B$ ($E_{\mathcal{G}}$ is full).

Another C*-algebra arising from a quantum graph

Definition

The Cuntz–Pimsner algebra $\mathcal{O}_X$ of a C*-correspondence X is the C*-algebra generated by a universal *Toeplitz covariant representation* of X.

(B, ψ, A) - quantum graph $\quad E_{\mathcal{G}}$ quantum edge correspondence

Theorem (Brannan-Hamidi-I.-Nelson-Wasilewski (2023))

When $\mathcal{G}$ has no quantum sources, $L\mathcal{O}(\mathcal{G}) \cong \mathcal{O}_{E_{\mathcal{G}}}$.

This realization of $L\mathcal{O}(\mathcal{G})$ as $\mathcal{O}_{E_{\mathcal{G}}}$ for lots of $\mathcal{G}$ is a key ingredient to showing distinction from $\mathcal{O}(\mathcal{G})$.

Another C*-algebra arising from a quantum graph

Theorem (Hamidi-I.-Nelson (2025))

When $\mathcal{G}$ has no quantum sources, $\mathcal{O}(\mathcal{G})$ surjects onto a relative Cuntz–Pimsner algebra $\mathcal{O}(K, E_{\mathcal{G}})$, *where* $K = B \cdot A(B) \cdot B$.

Corollary

When $\mathcal{G}$ has no quantum sources but a nontrivial set of quantum sinks (K is nontrivial), the canonical surjection $\mathcal{O}(K, E_{\mathcal{G}}) \to \mathcal{O}_{E_{\mathcal{G}}}$ is not injective. In particular, $\mathcal{O}(\mathcal{G})$ is not simple.

Simplicity of local quantum Cuntz–Krieger algebras

So far we have seen a few examples of quantum graphs whose local quantum Cuntz–Krieger algebras are simple:

- $K(B, \psi)$ by explicit isomorphism with a Cuntz algebra
- (M_n, ψ, A) where the span of Kraus operators for A has dimension >1 by showing Morita equivalence to a Cuntz algebra.

We can also use the Cuntz–Pimsner algebra realization of $L\mathcal{O}(\mathcal{G})$ to prove

- $L\mathcal{O}(B, \psi, \mathrm{id})$ is non-simple by Schweizer's simplicity conditions.
- $L\mathcal{O}(\mathcal{G}_0)$ is simple using Condition (S) from Eryüzlü, et. al.

Corollary

When $\mathcal{G}$ has no quantum sources but a nontrivial set of quantum sinks (K is nontrivial), the canonical surjection $\mathcal{O}(K, E_{\mathcal{G}}) \to \mathcal{O}_{E_{\mathcal{G}}}$ is not injective. In particular, $\mathcal{O}(\mathcal{G})$ is not simple.

A graph which separates $\mathcal{O}(\mathcal{G})$ and $L\mathcal{O}(\mathcal{G})$

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Example

There exists a quantum graph $\mathcal{G}_0$ which has no quantum sources, a nontrivial quantum sink, and whose edge correspondence satisfies the conditions for simplicity of $\mathcal{O}_{E_{\mathcal{G}}}$. Details on arXiv. Since $\mathcal{O}(\mathcal{G})$ is non-simple, it is not isomorphic to $L\mathcal{O}(\mathcal{G})$.

Thank you!