A Generalization of the Stone-von Neumann Theorem

October 25, 2025

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Quantum mechanics in the Hilbert space formalism

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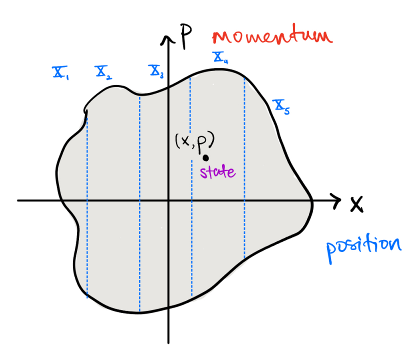

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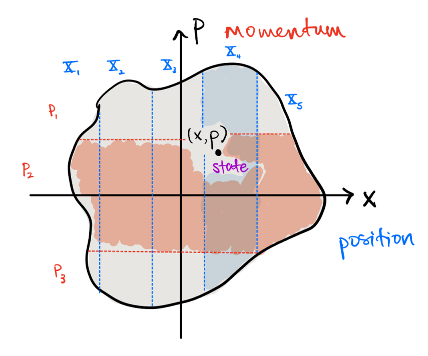

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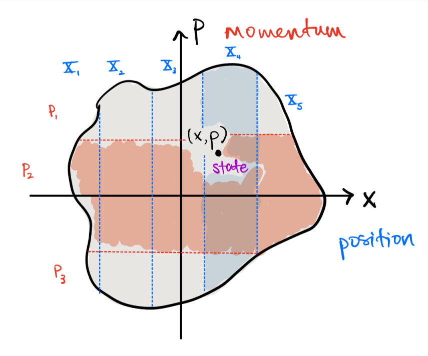

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Quantum mechanics formalism on Hilbert space

For quantum system, knowing x and p simultaneously is not possible.

$$\text{phase space} \rightsquigarrow \text{Hilbert space } \mathcal{H}$$

$$\text{state } (x, p) \rightsquigarrow \text{unit vector } \psi \in \mathcal{H}$$

$$\text{observable} \rightsquigarrow \text{self-adjoint operator } A \in B(\mathcal{H})$$

Example

	Classical	Quantization
Phase Space	$\cup_{i=1}^{m} X_i = \cup_{j=1}^{n} P_j$	$\vee_i \mathrm{Span}(X_i) = \vee_j \mathrm{Span}(P_j)$.
Position	$(x, p) \mapsto \sum_i x_i \cdot \chi_{X_i}(x)$	$\psi \mapsto \sum_i x_i \mathrm{Proj}_{X_i}(\psi)$
Momentum	$(x, p) \mapsto \sum_j p_j \cdot \chi_{P_j}(p)$	$\psi \mapsto \sum_j p_j \mathrm{Proj}_{P_j}(\psi)$

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Stone's Question

Consider P and X on $L^2(\mathbb{R})$ defined on $f \in \mathcal{C}$ by

$$(Pf)(x) = -if'(x) \quad \text{and} \quad (Xf)(x) = xf(x) \quad \forall x \in \mathbb{R}.$$

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One of these directions is very subtle.

The unitary groups associated to X and P

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Let G be a *locally compact abelian group*, and let $\widehat{G}$ be its *Pontryagin dual*. Two unitary group representations $U : G \to \mathcal{U}(\mathcal{H})$ and $V : \widehat{G} \to \mathcal{U}(\mathcal{H})$ form a Heisenberg representation of G if

$$V_\gamma U_t = \overline{\gamma(t)} U_t V_\gamma \quad t \in G, \gamma \in \widehat{G}.$$

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Example

Every G has a left regular representation. Here, $\mathcal{H} = L^2(G, \mu)$ (μ is Haar measure) and $\lambda : G \to \mathcal{U}(L^2(G))$ is given on $f \in \mathcal{C}$ by

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Then $\widehat{G} \to \mathcal{U}(\mathcal{H})$ given by $\gamma \mapsto m_\gamma$ is a unitary group representation.

Heisenberg representations on Hilbert space

Let G be a locally compact abelian group.

Definition

A Heisenberg representation of G is a triple $(\mathcal{H}, U, V)$ consisting of a Hilbert space $\mathcal{H}$, a unitary group rep $U : G \to \mathcal{U}(\mathcal{H})$, and a unitary group rep $V : \widehat{G} \to \mathcal{U}(\mathcal{H})$ such that

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The **von Neumann Uniqueness Theorem** says $(L^2(G), \lambda, m)$ is the *only* (irreducible) Heisenberg representation for G (up to equivalence).

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Theorem (Stone-von Neumann, 1931)

Every (irreducible) Heisenberg representation of G is (unitarily) equivalent to the Schrödinger representation:

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When a commutation relation for a pair (A, B) of self-adjoint operators "integrates" to a their unitary groups is studied in Lie theory.

Reframing in C*-algebras

Denote by $\widehat{F} : L^1(\widehat{G}) \to C_0(G)$ the *Fourier transform* for $\widehat{G}$. Then:

$$\left\{\text{unitary reps. of } \widehat{G}\right\} \longleftrightarrow \{C^*\text{-reps. of } C_0(G)\}.$$

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For groups: A l.c. abelian group G also gives rise to a C*-algebra $C^*(G)$:

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Motivation from Topology: Given a l.c. Hausdorff space X, consider C*-algebra $C_0(X)$ (continuous functions on X which "vanish at ∞") where $f^*(x) = \overline{f(x)}$, and whose norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$ satisfies

$$\|f^* f\|_\infty = \|f\|_\infty^2. \quad (C^*\text{-identity})$$

Banach-Stone: $X \cong Y$ if and only if $C_0(X) \cong C_0(Y)$ as C*-algebras.

For groups: A l.c. abelian group G also gives rise to a C*-algebra $C^*(G)$:

$$C^*(G) \cong C_0(\widehat{G}),$$

where $f \mapsto \widehat{f}(\gamma) = \int_G f(t) \overline{\gamma(t)} \, d\mu(t)$ extends to $\overline{L^1(G)} =: C^*(G)$.

Reframing in C*-algebras

C*-dynamical system: For $f \in C_0(G)$ and $s \in G$, define $(lt_s f)(t) = f(t-s)$ for all $t \in G$. Then

- $\mathrm{lt} : G \to \mathrm{Aut}(C_0(G))$ is a cts. group h'ism, and
- $(C_0(G), G, lt)$ is a *C*-dynamical system*.

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Representations: Define $\pi : C_0(G) \to B(L^2(G))$ by $\pi(f)g = fg$ for all $g \in L^2(G)$. Then for all $f \in C_0(G)$,

$$\lambda_s \circ \pi(f) = \pi(\mathrm{lt}_s f) \circ \lambda_s \quad \forall s \in G.$$

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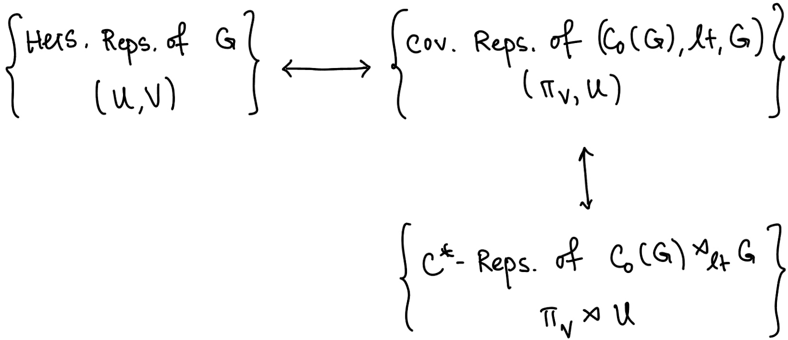
$$(L^2(G), \lambda, m) \longleftrightarrow (\pi_m, \lambda).$$

Proof Outline

$$\left\{ \begin{array}{c} \text{Heis. Reps. of } G \\ (U, V) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Cov. Reps. of } (C_0(G), lt, G) \\ (\pi_V, U) \end{array} \right\}$$

Proof Outline



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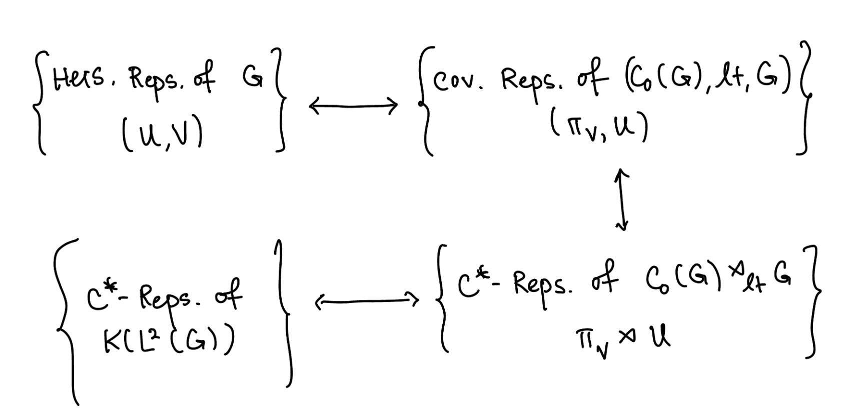

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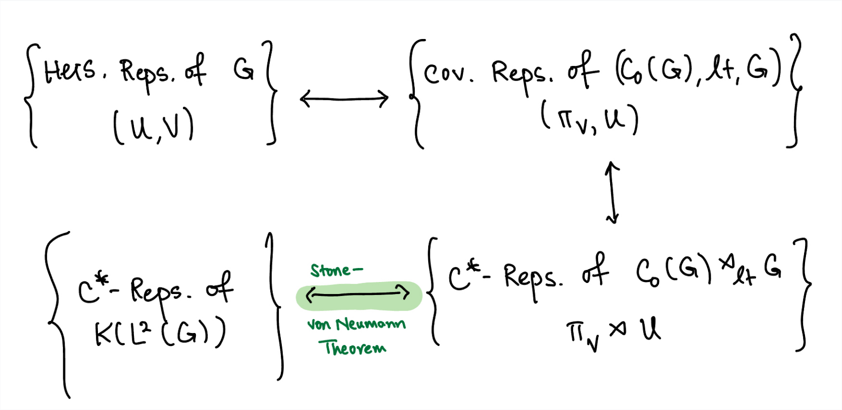

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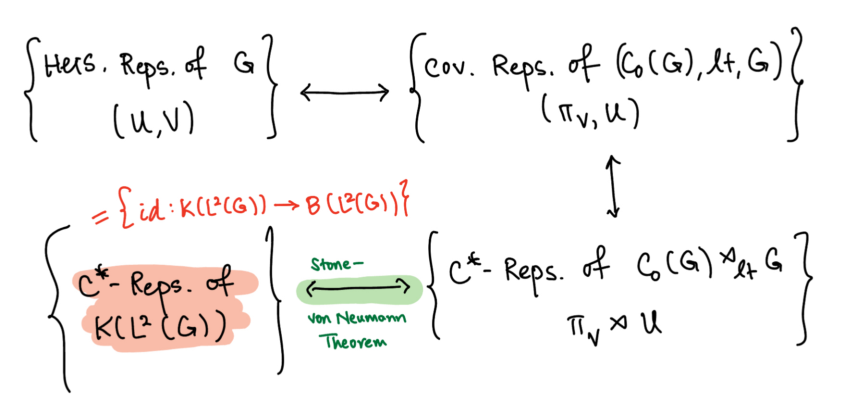

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Every (irreducible) Heiseberg representation $(\mathcal{H}, U, V)$ of G is (unitarily equivalent to) the Schrödinger rep $(L^2(G), \lambda, m)$.

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Can classify more general notions of Heisenberg pairs?

- Hilbert space formalism is restricted to non-relativistic quantum mechanics
- May be more symmetries of the quantum system we want to encode

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A Hilbert C^*-module is a bimodule X over a C*-algebra B equipped with a complete inner product $\langle \cdot \mid \cdot \rangle_B$ which takes values in B, and

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- $\langle x \mid y \cdot b \rangle_B = \langle x \mid y \rangle_B b$
- $\langle x \mid y \rangle_B^* = \langle y \mid x \rangle_B$
- $\langle x \mid a \cdot y \rangle_B = \langle a^* \cdot x \mid y \rangle_B$

for all $a, b \in B$ and $x, y \in X$.

Example

1. When $B = \mathbb{C}$, X is a Hilbert space.
2. We can consider $M_n(\mathbb{C})$ as a bimodule over itself using matrix multiplication, and define

$$\langle ((x_{ij})_{i,j} \mid (y_{ij})_{i,j} \rangle_{M_n} := (\sum_k \overline{x_{ki}} y_{kj}))_{i,j}$$

Definition

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Interpretation:

- Hilbert C*-modules allow us to handle many-body quantum systems.
- Representations of C*-dynamical systems, as opposed to only unitary group representations, allow us to endow our quantum system with quantum symmetries (rather than those afforded by only a group's classical symmetries).

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Theorem (Huang-I, 2020)

Every representation on a Hilbert C^*-module of a C^*-dynamical system of locally compact abelian group acting on the C^*-algebra of compact operators is equivalent to a canonical Schrödinger representation.

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Tools:

1. We have a Stone-von Neumann isomorphism of a C*-algebra arising from a canonical dynamical system with the compact operators on our canonical Hilbert C*-module $(C_0(G) \rtimes_{lt} G \cong K(L^2(G))$.

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2. We showed all representations of the compact operators on our canonical Hilbert C^*-module were equivalent to the identity representation.

Questions

- Nonabelian groups?
- Other Hilbert C^*-modules?
- Quantum groups?
- Integrability?

Thank you!