# Simplicity of Cuntz–Pimsner algebras of quantum graphs

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# $C^*$-algebras associated to graphs

Let $G = (V, A)$ be a finite simple graph ($A \in M_n(\{0,1\})$ for $n = |V|$).

### *Definition*

A Cuntz–Krieger G-family is a linear map $S : C(V) \to \mathcal{D}$ satisfying

- $S_i S_i^* S_i = S_i \quad \forall i \in V$
- $S_i^* S_i = \sum_{j \in V} A_{ij} S_j S_j^* \quad \forall i \in V$
- $\sum_{i \in V} S_i S_i^* = 1$

The Cuntz–Krieger algebra $\mathcal{O}_A$ is the universal $C^*(S)$.

### *Theorem (Rørdam '94)*

*Suppose A has no identically zero rows or columns.*
**When $\mathcal{O}_A$ is simple**, *$K_0(\mathcal{O}_A)$ is a complete invariant.*

# The quest for a quantum analogue

To be defined: Let $G$ be a *finite simple quantum graph*, and let $S$ be a *quantum Cuntz–Krieger $G$-family*.

- There is a (free) quantum Cuntz–Krieger algebra $\mathbb{F}\mathcal{O}(G)$ (Brannan-Eifler-Voigt-Weber '22).
- There is a unital quantum Cuntz–Krieger algebra $\mathcal{O}(G)$, which is quotient of $\mathbb{F}\mathcal{O}(G)$ by a natural unit condition.
- There is a "local" quantum Cuntz–Krieger algebra $L\mathcal{O}(G)$, which is a quotient of $\mathcal{O}(G)$ (Brannan-Hamidi-I.-Nelson-Wasilewski'23).

***Theorem (Brannan-Hamidi-I.-Nelson-Wasilewski '23)***

*Let $G$ be a finite simple quantum graph. When $G$ has no* quantum sources *the Cuntz–Pimsner algebra for the quantum edge correspondence $E_G$ is* $\cong L\mathcal{O}(G)$.

**Starting Goal:** Characterize simplicity of $L\mathcal{O}(G) \cong O_{E_G}$.

# Finite quantum sets

*Definition*

A finite quantum set is a pair $(B, \psi)$ where

- $B$ is a finite-dimensional $C^*$-algebra, written $B \cong \bigoplus_{a=1}^{d} M_{n(a)}$,
- $\psi$ is a faithful state on $B$ (corresponding to a unique density matrix $Q \in B$)

There are canonical maps

- $m : L^2(B \otimes B) \to L^2(B)$ by $e_{ij}^{(a)} \otimes e_{k\ell}^{(b)} \mapsto \delta_{j=k}\delta_{a=b}\, e_{i\ell}$, and
- $m^* : L^2(B) \to L^2(B \otimes B)$ by $e_{ij}^{(a)} \mapsto \sum_k Q_{a,k}^{-1} e_{ik}^{(a)} \otimes e_{kj}^{(a)}$.

We require that $\psi$ be a $\delta$-form: $mm^* = \delta^2 \mathrm{id}_B \iff \mathrm{Tr}(Q_a^{-1}) = \delta^2 \; \forall a$

# Examples of quantum sets

*Example*

$V$ a finite set, $B = C(V)$ and $\psi(e_i) = |V|^{-1}$ for all $i \in V$. Here,

$$m(e_i \otimes e_j) = \delta_{i=j} e_i \text{ and } m^*(e_i) = |V| (e_i \otimes e_i),$$

so $mm^* = |V| \, \mathrm{id}_{C(V)}$.

*Example*

$B = M_n$ and $\psi(x) = \mathrm{Tr}(x)$ for all $x \in M_n$. Here, $mm^* = n^2 \mathrm{id}_{M_n}$.

***Theorem***

*There is a unique tracial δ-form on any* $B \cong \bigoplus_{a=1}^{d} M_{n(a)}$.

# Finite quantum graphs

### *Definition (Brannan-Eifler-Voigt-Weber'22)*

A finite quantum graph is a triple $G = (B, \psi, A)$ where

- $(B, \psi)$ is a finite quantum graph such that $\psi$ is a $\delta$-form, and
- $A : B \to B$ is a completely positive *quantum adjacency matrix*:

$$m(A \otimes A)m^* = \delta^2 A.$$

### *Definition (Weaver '21)*

A quantum graph is an operator system $S \subseteq \mathcal{M}$ inside a von Neumann algebra $\mathcal{M}$ which is a bimodule over $\mathcal{M}'$.

### *Proposition (Daws '24)*

*When $\mathcal{M}$ is finite-dimensional and $A$ is self-adjoint and* reflexive, *these notions coincide.*

# Examples of quantum graphs

1. Complete: $K(M_n, \mathrm{Tr})$

Adjacency Matrix (in basis $e_{ij}$):
$A(x) = n^2 \mathrm{Tr}(x) 1 \quad \forall x \in M_n$

$$\begin{bmatrix} n & 0 & \dots & 0 \\ 0 & \ddots & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & n \end{bmatrix}$$

Operator System:
$S = M_n$

$$\begin{bmatrix} * & * & \dots & * \\ * & \ddots & & * \\ \vdots & & \ddots & \vdots \\ * & \dots & \dots & * \end{bmatrix}$$

2. Trivial: $T(M_n, \mathrm{Tr})$
$A(x) = x \quad \forall x \in M_n$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix}$$

Operator System: $S = \mathbb{C}I_n$

$$\begin{bmatrix} x & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & x \end{bmatrix}$$

# Quantum edge correspondence $E_G$

Define $B \otimes_\psi B$ to be the $B$-correspondence with usual left/right action, and $B$-valued inner product: $\langle x_1 \otimes y_1 \mid x_2 \otimes y_2 \rangle = \psi(x_1^* x_2) y_1^* y_2 \; \forall x_i, y_i \in B$.

## *Definition*

Let $G = (B, \psi, A)$ be a quantum graph.

- Define $\varepsilon_G := \delta^{-2}(\mathrm{id}_B \otimes A)m^*(1)$.
- The quantum edge correspondence $E_G$ is the $B$-subcorrespondence of $B \otimes_\psi B$ spanned by $\{x \cdot \varepsilon_G \cdot y : x, y \in B\}$.

## *Example*

Let $(V, A)$ be a classical graph, so $(C(V), |V|^{-1}, A)$ is a quantum graph. Then

$$\epsilon_G = (\mathrm{id} \otimes A)\left(\sum_i e_i \otimes e_i\right) = \sum_{i,j} A_{ij} e_i \otimes e_j.$$

As a function in $C(V) \otimes C(V) \cong C(V \times V)$, we have $\epsilon_G = \chi_E$

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As a function in $C(V) \otimes C(V) \cong C(V \times V)$, we have $\epsilon_G = \chi_E$ and $E_G \cong C(E)$.

# Edges for $K(M_2, \mathrm{Tr})$

Recall the complete graph $K := K(M_2, \mathrm{Tr})$ has adjacency matrix

$$A(x) = 4\mathrm{Tr}(x)1; \quad \forall x \in M_2.$$

Then $\varepsilon_K = (\mathrm{id} \otimes \mathrm{Tr}(\cdot)1)m^*(1) = 1 \otimes 1$, and $E_K$ is the $M_2$-subcorrespondence of $M_2 \otimes_{\mathrm{Tr}} M_2$ generated by $\varepsilon_T$:

$$E_K = \mathrm{Span}\{x \cdot (1 \otimes 1) \cdot y : x, y \in M_2\} = M_2 \otimes M_2.$$

The picture I might draw for this edge correspondence is $K_4$.

### *Theorem (Brannan-Hamidi-I.-Nelson-Wasilewski '23)*

*For any finite set $(B, \psi)$, the local quantum Cuntz–Krieger algebra $L\mathcal{O}(K(B, \psi))$ is isomorphic to the Cuntz algebra on $\dim B$ generators.*

$L\mathcal{O}(K(B, \psi))$ is simple if and only if $\dim B > 1$.

# Local quantum Cuntz–Krieger algebras

Let $G = (B, \psi, A)$ be a quantum graph such that $\psi$ is a $\delta$-form and $A$ is cp. For a unital $C^*$-algebra $\mathcal{D}$, let $\mu : \mathcal{D} \odot \mathcal{D} \to \mathcal{D}$ denote the map $x \otimes y \mapsto xy$.

## *Definition*

A local quantum Cuntz–Krieger G-family is a linear map $S : B \to \mathcal{D}$ which satisfies

1. $\mu(\mathrm{id} \otimes \mu)(S \otimes S^* \otimes S)(m^* \otimes \mathrm{id}) = \delta^{-2} Sm$
2. $\mu(S^* \otimes S) = \delta^{-2}\mu(S \otimes S^*)m^*Am$
3. $\mu(S \otimes S^*)m^*(1_B) = \delta^{-2} 1_{\mathcal{D}}$.

When $B = M_n$ and $S_{ij} := nS(e_{ij})$, these relations are equivalent to:

1. $\sum_k S_{\ell k} S^*_{mk} S_{pq} = \frac{1}{n}\delta_{m=p} S_{\ell q}$
2. $\sum_k S^*_{k\ell} S_{km} = \delta^{-2}\mu(S \otimes S^*)m^*Am$
3. $\sum_{\ell,m} S_{\ell m} S^*_{\ell m} = 1_{\mathcal{D}}$

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2. $\mu(S^* \otimes S) = \delta^{-2} \mu(S \otimes S^*) m^* A m$
3. $\mu(S \otimes S^*) m^*(1_B) = \delta^{-2} 1_{\mathcal{D}}$.

## *Definition*

The local quantum Cuntz–Krieger algebra $L\mathcal{O}(G)$ for $G$ is the universal $C^*$-algebra generated by such a family.

**Remark:** Computing $L\mathcal{O}(G)$ is hard, let alone $\mathcal{O}(G)$ or $\mathbb{F}\mathcal{O}(G)$. Its isomorphism with $\mathcal{O}_{E_G}$ helps, but challenge remains nonetheless.

# Roadmap for simplicity results

We apply three distinct approaches:

1. The single-vertex (single matrix algebra) case using a result by Marrero–Muhly.
2. The full-correspondence case: Schweizer's criterion (minimality + aperiodicity).
3. The non-full-correspondence case: Condition (S) (Eryüzlü et. al.)

# Kraus Decompositions for Completely Positive Maps

Recall: If $\Phi(\cdot) = \sum_\ell K_\ell(\cdot)K_\ell^*$ for some $\{K_\ell\} \in M_n$,

- then $\Phi$ is cp (and conversely, $\Phi$ cp implies it has such a decomposition)
- the subspace Span$\{K_\ell\}$ of $M_n$ is unique (even if choice of $\{K_\ell\}$ is not)
- the dimension $d(\Phi)$ of Span$\{K_\ell\}$ is well-defined

## *Theorem (Marrero–Muhly '06)*

*Let $\Phi : M_n \to M_n$ be completely positive. Define the $M_n$-correspondence $M_n \otimes_\Phi M_n$, defined as $M_n \otimes M_n$ with usual left/right actions and*

$$\langle x_1 \otimes y_1 \mid x_2 \otimes y_2 \rangle = \Phi(x_1^* x_2) y_1^* y_2.$$

*Then the Cuntz–Pimsner algebra for $M_n \otimes_\Phi M_n$ is Morita equivalent to the Cuntz algebra on $d(\Phi)$ generators.*

We tend to require that our quantum adjacency matrices are cp (equivalent to being $*$-preserving for these operators).

# Application to local quantum Cuntz–Krieger algebras

### *Proposition (Brannan-Hamidi-I.-Nelson-Wasilewski '23)*

*For* $G = (M_n, Tr, A)$ *we have* $E_G \cong M_n \otimes_A M_n$.

### *Theorem (Hamidi-I.-Nelson '26)*

*Let* $G = (M_n, Tr, A)$. *Then* $L\mathcal{O}(G)$ *is simple if and only if* $d(A) > 1$, *and in particular, if and only if* $A(x) = TxT^*$ *for* $T \in M_n$ *which satisfies* $Tr(Q^{-1}T^*T) = n^2$.

### *Example*

The trivial graph (any automorphism graph) gives rise to non-simple $L\mathcal{O}(G)$.

### *Example*

Given any orthogonal (in trace) collection $\{K_\ell\} \subseteq M_n$ of cardinality greater than 1, the cp map $A(\cdot) := \sum_\ell K_\ell(\cdot)K_\ell^*$ is a quantum adjacency matrix with $d(A) > 1$, so $L\mathcal{O}(M_n, \mathrm{Tr}, A)$ is simple.

# Characterization of simplicity for full $C^*$-correspondences

**Definition**

A $B$-correspondence $E_G$ is full if the ideal $\langle E_G \mid E_G \rangle$ is dense in $B$.

**Proposition (Brannan-Hamidi-I.-Nelson-Waslewski '23)**

*The quantum edge correspondence for $G = (B, \psi, A)$ is full if and only if the ideal $B \cdot A(B) \cdot B$ is all of $B$. We say "G has no quantum sinks."*

**Theorem (Schweizer '01)**

*If $E_G$ is full, then $\mathcal{O}_{E_G}$ is simple $\iff$ $E_G$ is minimal and aperiodic.*

**Proposition (Hamidi-I.-Nelson '26)**

*Suppose $E_G$ is full.*

- *$E_G$ is minimal $\iff$ there is no nontrivial ideal $J \lhd B$ with $A(J) \subseteq J$.*
- *If $E_G$ is not faithful ("has quantum sources"), then $E_G$ is aperiodic.*

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Ongoing work...

- Determine simplicity of $L\mathcal{O}(G)$ for more quantum graphs in terms of $E_G$ using Eryüzlü et. al.'s Condition (S)
- Compute $K_0(L\mathcal{O}(G))$ for simple $L\mathcal{O}(G)$.
- Formalize visualization tools for $G$ which give rise to properties of $L\mathcal{O}(G)$.
- Compute more examples of $L\mathcal{O}(G)$!

Thank you!